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Estimation the Reliability of the Triple Modular Redundancy system by Using the Fréchet Distribution

Husam kamel Issa [1] , Mustafa Abdullah Jaralah [2] , Ghaith rahman thuaban [3]

Abstract

Triple modular redundancy (TMR) system is a powerful technique used for reducing the errors and failures that occur in the backup system because it increases the reliability of the systems, as these systems consist of three components. These components are exposed to progressive life tests higher than the normal pressure. Important mathematical properties of the system distribution were derived as a probability density function (PDF), cumulative density function (CDF), system reliability and risk function. This study focused on comparing three estimation methods (maximum likelihood, least squares error and maximum product of spacing) for the "TMR" system under the gradually accelerating life test that follows the Fréchet distribution. To compare the estimation methods, the comparison criterion of the adjusted mean square error is used, as Monte Carlo simulation was studied. This research aims to find out the most efficient method among the three estimation methods for this triple modular system. The most important conclusions in this study are that the maximum likelihood method is the best in estimation and more efficient and has the lowest adjusted mean square error compared to the other method.

Keywords: TMR, Frechet Distribution, Simulation, MLE, LSE.MPS

Affiliation of Authors

[1, 2, 3] Department of Accounting College of administration and Economics, University of kut, Iraq, Wasit, 52001

[1] hossam.Issa@uokut.edu.iq
[2] mustafa.abdullah@uokut.edu.iq
[3] ghaith.rahman@uokut.edu.iq

[1] **Corresponding Author**

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تقدير معولية نظام الاحتياطي الثلاثي المعياري باستعمال توزيع فريشت

م. م. حسام كامل عيسى [1] ، م. م. مصطفى عبدالله جار الله [2] ، م. م. غيث رحمان ثعبان [3]

المستخلص

يُعد نظام التكرار الثلاثي المعياري (TMR) تقنية فعّالة تُستخدم لتقليل الأخطاء والأعطال التي تحدث في نظام التكرار الاحتياطي، وذلك لأنه يزيد من معولية الأنظمة، حيث تتكون هذه الأنظمة من ثلاثة مكونات. اذ تتعرض هذه المكونات لاختبارات الحياة المتسارعة تدرجياً وتكون أعلى من الضغط الطبيعي. وقد تم اشتقاق خصائص رياضية مهمة لتوزيع النظام، مثل دالة كثافة الاحتمال (PDF)، ودالة الكثافة التراكمية (CDF)، ودالة معولية النظام، ودالة المخاطر. و ركزت هذه الدراسة على مقارنة ثلاث طرق تقدير (الامكان الاعظم ، و طريقة المربعات الصغرى، وطريقة أعظم ناتج للفروق) لنظام "TMR" في ظل اختبار عمري متسارع تدريجيًا الذي يتبع توزيع فريشة. وللمقارنة بين طرائق التقدير، تم استخدام معيار متوسط مربع الخطأ المعدل، حيث تمت دراسة محاكاة مونت كارلو. يهدف هذا البحث إلى إيجاد الطريقة الأكثر كفاءة من بين طرق التقدير الثلاث لهذا النظام الثلاثي المعياري. أهم الاستنتاجات في هذه الدراسة هي أن طريقة الامكان الاعظم هي الأفضل في التقدير وأكثر كفاءة ولها أقل متوسط خطأ تربيعي معدل مقارنة بالطريقة الأخرى.

الكلمات المفتاحية: توزيع فريشت، نظام الاحتياطي الثلاثي، المحاكاة، طريقة مربعات الصغرى، طريقة الإمكان الأعظم، طريقة اعظم ناتج للفروق

انتساب الباحثين

[1، 2، 3] قسم المحاسبة، كلية الإدارة والاقتصاد، جامعة الكوت، العراق، واسط، 52001

[1] hossam.Issa@uokut.edu.iq
[2] mustafa.abdullah@uokut.edu.iq
[3] ghaith.rahman@uokut.edu.iq

[1] **المؤلف المراسل**

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Introduction

TMR is a powerful technique that relies on the redundancy of important components within the system. Systems that have a redundancy of three times the basic components are referred to as triple standard redundancy (TMR) systems [1]. The primary goal of the systems is to enhance reliability and reduce the failure rate of the system. It is very important for these systems to maintain a constant level of operation even in the event of a minor system failure Moreover, these systems must maintain a consistent level of performance. With ongoing advancements in manufacturing design and technology, components have become more durable and exhibit lower failure rates, which increases their overall significance. Consequently, conducting life tests under normal operating conditions has become increasingly challenging, as such tests require substantial time and financial resources. For this reason, conventional life testing methods are often impractical, particularly during the development of new component prototypes. A practical solution to this challenge is the use of Accelerated Life Tests (ALTs), in which test units are exposed to higher-than-normal stress levels to induce failures more quickly. ALT enables researchers to obtain sufficient failure data within a shorter timeframe and to better understand the relationship between product lifetime and external stress factors. In (2016), the researchers compared the reliability of the triple and five-standard reserve systems [3]. In (2017), the researchers estimated the failure function and reliability using the Weibull distribution for the triple-standard reserve system [4]. In (2021), the researchers used the exponential distribution with type I controlled data to calculate the failure rate in gradually accelerated life tests [5]. In (2021), the researcher used the Exponential-Poisson distribution to analyze the reliability of the continuous stress gradual life test model [6]. In (2023), the researcher used the Lomax distribution to estimate the triple-standard reserve system in the presence of gradually accelerated life tests [7]. In (2024), the researchers estimated parameters using the Frechette distribution for the accelerated life tests of the triple-standard reserve system [8]. The study aims to estimate the reliability of the (TMR) system under gradually accelerated life tests using the Fréchet distribution. By using three methods (least squares method, maximum likelihood method, and maximum product of successive differences method), using the Monte Carlo simulation experiments method and to compare the three estimation methods using the average square error rate to choose the best method.

1. Fréchet distribution

It is one of the continuous distributions presented by the French mathematician (Maurice-Fréchet) and has wide applications in interpreting events such as life tests like as the rainfall and so on. It is used in interpreting and creating error models and modeling failure rates that are used in many studies, including reliability and biological studies. Where θ> 0 and α> 0.The $F(t)$ and $f(t)$ and $R(t)$ and $h(t)$ are given by [9],[10]:

$$f(\mathbf{t};\boldsymbol{\theta},\boldsymbol{\alpha}) = \frac{\boldsymbol{\alpha}}{\mathbf{t}}\left(\frac{\boldsymbol{\theta}}{\mathbf{t}}\right)^{\alpha} e^{-\left(\frac{\theta}{t}\right)^{\alpha}} \tag{1}$$

$$\mathbf{F(t)} = e^{-\left(\frac{\theta}{t}\right)^{\alpha}} \tag{2}$$

$$\mathbf{R(t)} = 1 - e^{-\left(\frac{\theta}{t}\right)^{\alpha}} \tag{3}$$

$$\mathbf{h(t)} = \frac{\frac{\alpha}{t}\left(\frac{\theta}{t}\right)^{\alpha} e^{-\left(\frac{\theta}{t}\right)^{\alpha}}}{1 - e^{-\left(\frac{\theta}{t}\right)^{\alpha}}} \tag{4}$$

2. K-out of-N systems

It is a system whose construction depends on multiple subsystems that may be linked together in a serial or parallel manner, and it does not prevent the failure of a certain number of subsystems due to the existence of independent and effective subsystems despite the similarity of these systems [11], [12].

$$R_k - \mathrm{out} - \mathrm{of} - \mathrm{N(t)} = \sum_{k=i}^{n} \binom{n}{k} [R(t)]^k \, [1 - R(t)]^{n-k} \qquad (5)$$

3. Reliability of the triple modular redundancy system

This system consists of three components, A1, A2, A3. These components are independent of each other, and are connected to the TMR system. At least two of the three components must be working for this system to continue the working. Assuming a random variable (T), (t) represents the lifetime of the (TMR) system and the reliability of the components is equal to each other and independent as shown in the following equations [1], [8].

$$R_{TMR} = R_{A1(t)} R_{A2}\left(1 - R_{A3(t)}\right) + R_{A1(t)}\, R_{A3}\left(1 - R_{A2(t)}\right) + R_{A3(t)}\, R_{A2}\left(1 - R_{A1(t)}\right) + R_{A1(t)}\, R_{A2(t)}\, R_{A3(t)}$$

$$R_{Ai(t)} = R(t) \qquad i = 1, 2, 3$$

By using n=2, K=3 in equation (5), then the result will be the ($R_{TMR(t)}, F_{TMR(t)}, f_{TMR(t)}, h_{TMR(t)}$).

The statistical properties to any distribution are supposed some relationships between the some of distribution functions like as the relationship between reliability function and (CDF) function, from this step it is possible to get the below functions.

$$R_{TMR(t)} = 3R^2(t) - 2R^3(t) \qquad (6)$$

$$F_{TMR(t)} = 2R^3(t) - 3R^2(t) + 1 \qquad (7)$$

$$f_{TMR(t)} = 6f_{(t)}[1 - R(t)]R(t) \qquad (8)$$

$$F_{TMR(t)} = 3\, e^{-2\left(\frac{\theta}{t}\right)^{\alpha}} - 2e^{-3\left(\frac{\theta}{t}\right)^{\alpha}} \qquad (9)$$

$$f_{TMR(t)} = 6\, \frac{\alpha}{t} \left(\frac{\theta}{t}\right)^{\alpha} \left[e^{-2\left(\frac{\theta}{t}\right)^{\alpha}} - e^{-3\left(\frac{\theta}{t}\right)^{\alpha}}\right] \qquad (10)$$

$$R_{TMR(t)} = 1 - 3\, e^{-2\left(\frac{\theta}{t}\right)^{\alpha}} + 2e^{-3\left(\frac{\theta}{t}\right)^{\alpha}} \qquad (11)$$

$$h_{TMR(t)} = \frac{6\, \frac{\alpha}{t} \left(\frac{\theta}{t}\right)^{\alpha} \left[e^{-2\left(\frac{\theta}{t}\right)^{\alpha}} - e^{-3\left(\frac{\theta}{t}\right)^{\alpha}}\right]}{1 - 3\, e^{-2\left(\frac{\theta}{t}\right)^{\alpha}} + 2e^{-3\left(\frac{\theta}{t}\right)^{\alpha}}} \qquad (12)$$

4. Step-Partially Accelerated life tests

The technique of step stress partial accelerated (ALT) is applied in this part to the TMR system, assuming that T is a random variable with transformation functions ($F_{(t)}$ and $f_{(t)}$ $h_{(t)}$, $R_{(t)}$) as shown in the following equations (9,10,11,12). Two techniques are used in this system to increase the step stress, one of which is the controlling failure rate and the other is the controlling random variable. When the specified time is reached, the variable (η) increases the pressure directly and remains in a stable state, and the failure rate increases by (β>1) and by multiplying the risk function of the TMR system by (β>1) as in the following equations [2], [13].

$$\mathbf{h^*(t)} = \begin{cases} \mathbf{h_{TMR(t)}} & \mathbf{0 < t \le \eta} \\ \mathbf{\beta h_{TMR(t)}} & \mathbf{\eta < t < \infty, (\beta > 1)} \end{cases} \qquad (13)$$

It is possible to calculate the reliability of $R^*_{(t)}$ as below:

$$\boldsymbol{R^*_{(t)}} = \begin{cases} \boldsymbol{R_{1(t)}} = \mathbf{e}^{\left[-\int_0^t \mathbf{h\ TMR(u)\ du}\right]} & \boldsymbol{0 < t \le \eta} \\ \boldsymbol{R_{2(t)}} = \mathbf{e}^{\left[-\int_0^{\eta} \mathbf{h\ TMR(u) du} - \int_{\eta}^{t} \mathbf{\beta\ h\ TMR(u) du}\right]} & \boldsymbol{\eta < t < 0} \end{cases} \qquad (14)$$

From the equation (14) it can get the equation (15).

$$\mathbf{R_{2(t)}} = \mathbf{e}^{\left[-\int_0^{\eta} \frac{\mathbf{f\ TMR(u)}}{\mathbf{R\ TMR(u)}} \mathbf{du} - \int_{\eta}^{t} \mathbf{\beta} \frac{\mathbf{f\ TMR(u)}}{\mathbf{S\ TMR(u)}} \mathbf{du}\right]} = \left[\mathbf{R_{TMR(t)}}\right]^{\boldsymbol{\beta}} \left[\mathbf{R_{TMR(\eta)}}\right]^{\mathbf{1-\beta}} \qquad \mathbf{(15)}$$

By utilized equation (11) in equation (15) the result will be a R^* (t) $F^*(t)$, $f^*(t)$

$$\mathbf{R_{2(t)}} = \left[\mathbf{1 - 3\ e}^{-2\left(\frac{\theta}{t}\right)^{\alpha}} + \mathbf{2e}^{-3\left(\frac{\theta}{t}\right)^{\alpha}}\right]^{\boldsymbol{\beta}} * \left[\mathbf{1 - 3}\ \boldsymbol{e}^{-2\left(\frac{\theta}{\eta}\right)^{\alpha}} + \mathbf{2}\boldsymbol{e}^{-3\left(\frac{\theta}{\eta}\right)^{\alpha}}\right]^{\boldsymbol{1-\beta}} \qquad (16)$$

By utilizing the equations (11), (14) and (16), the outcome will be the (PALT).

$$\mathbf{R^*\ (t)} = \begin{cases} \mathbf{R_{1(t)}} = \left[\mathbf{1 - 3\ e}^{-2\left(\frac{\theta}{t}\right)^{\alpha}} + \mathbf{2e}^{-3\left(\frac{\theta}{t}\right)^{\alpha}}\right] & \mathbf{0 < t \le \eta} \\ \mathbf{R_{2(t)}} = \left[\mathbf{1 - 3\ e}^{-2\left(\frac{\theta}{t}\right)^{\alpha}} + \mathbf{2e}^{-3\left(\frac{\theta}{t}\right)^{\alpha}}\right]^{\boldsymbol{\beta}} * \left[\mathbf{1 - 3\ e}^{-2\left(\frac{\theta}{\eta}\right)^{\alpha}} + \mathbf{2e}^{-3\left(\frac{\theta}{\eta}\right)^{\alpha}}\right]^{\mathbf{1-\beta}} & \mathbf{\eta < t < \infty} \end{cases} \qquad (17)$$

$$\mathbf{F^*(t)} = \begin{cases} \boldsymbol{F_{1(t)}} = \mathbf{3}\ \boldsymbol{e}^{-2\left(\frac{\theta}{t}\right)^{\alpha}} - \mathbf{2}\boldsymbol{e}^{-3\left(\frac{\theta}{t}\right)^{\alpha}} & \boldsymbol{0 < t \le \eta} \\ \boldsymbol{F_{2(t)}} = \mathbf{1} - \left[\mathbf{1 - 3}\ \boldsymbol{e}^{-2\left(\frac{\theta}{t}\right)^{\alpha}} + \mathbf{2}\boldsymbol{e}^{-3\left(\frac{\theta}{t}\right)^{\alpha}}\right]^{\boldsymbol{\beta}} * \left[\mathbf{1 - 3}\ \boldsymbol{e}^{-2\left(\frac{\theta}{\eta}\right)^{\alpha}} + \mathbf{2}\boldsymbol{e}^{-3\left(\frac{\theta}{\eta}\right)^{\alpha}}\right]^{\boldsymbol{1-\beta}} & \boldsymbol{\eta < t < \infty} \end{cases} \qquad (18)$$

$$f^*(t) = \begin{cases} f_{1(t)} = 6\,\frac{\alpha}{t}\left(\frac{\theta}{t}\right)^{\alpha}\left[e^{-2\left(\frac{\theta}{t}\right)^{\alpha}} - e^{-3\left(\frac{\theta}{t}\right)^{\alpha}}\right] & 0 < t \le \eta \\ f_{2(t)} = 6\frac{\alpha}{t}\,\beta\left(\frac{\theta}{t}\right)^{\alpha}\left[e^{-2\left(\frac{\theta}{t}\right)^{\alpha}} - e^{-3\left(\frac{\theta}{t}\right)^{\alpha}}\right]\left[1 - 3\,e^{-2\left(\frac{\theta}{t}\right)^{\alpha}} + 2e^{-3\left(\frac{\theta}{t}\right)^{\alpha}}\right]^{\beta-1}\left[1 - 3\,e^{-2\left(\frac{\theta}{\eta}\right)^{\alpha}} + 2e^{-3\left(\frac{\theta}{\eta}\right)^{\alpha}}\right]^{1-\beta} & \eta < t < \infty \end{cases} \quad (19)$$

5. Maximum likelihood method (MLE)

MLE method is used in statistical inference because it contains many properties, including stability, consistency, and sufficiency. This method depends on maximizing the probability function. Assuming that x1, x2, x3 are random variables distributed according to the system and with three parameters, the estimator is the one that makes the maximum probability function at its maximum limit and is obtained by deriving the maximum probability function with respect to the three distribution parameters after taking its logarithm and setting it equal to zero [7], [15].

$$L(t;\theta,\alpha,\beta) \propto \left[\prod_{i=1}^{n_1} f_1(t_i)\right] * \left[\prod_{i=n_1+1}^{n} f_2(t_i)\right] \quad (20)$$

$$L(t;\theta,\alpha,\beta) \propto \left[\prod_{i=1}^{n_1}\frac{6\alpha}{t}\left(\frac{\theta}{t}\right)^{\alpha}\left(e^{-2\left(\frac{\theta}{t}\right)^{\alpha}} - e^{-3\left(\frac{\theta}{t}\right)^{\alpha}}\right)\right]\left[\prod_{i=n_1+1}^{n}\frac{6\alpha\beta}{t}\left(\frac{\theta}{t}\right)^{\alpha}\left(e^{-2\left(\frac{\theta}{t}\right)^{\alpha}} - e^{-3\left(\frac{\theta}{t}\right)^{\alpha}}\right)\left(1 - 3\,e^{-2\left(\frac{\theta}{t}\right)^{\alpha}} + 2e^{-3\left(\frac{\theta}{t}\right)^{\alpha}}\right)^{\beta-1} * \left(1 - 3\,e^{-2\left(\frac{\theta}{\eta}\right)^{\alpha}} + 2e^{-3\left(\frac{\theta}{\eta}\right)^{\alpha}}\right)^{1-\beta}\right] \quad (21)$$

$$L(t;\theta,\alpha,\beta) \propto (6\alpha)^{n1}(\theta)^{n\alpha}\prod_{i=1}^{n1}\frac{1}{t_i^{\alpha+1}}\prod_{i=1}^{n}\left(e^{-2\left(\frac{\theta}{t_i}\right)^{\alpha}} - e^{-3\left(\frac{\theta}{t_i}\right)^{\alpha}}\right)(\beta)^n\prod_{i=n1+1}^{n}\left(1 - 3\,e^{-2\left(\frac{\theta}{t_i}\right)^{\alpha}} + 2e^{-3\left(\frac{\theta}{t_i}\right)^{\alpha}}\right)^{\beta-1}\left(1 - 3\,e^{-2\left(\frac{\theta}{\eta}\right)^{\alpha}} + 2e^{-3\left(\frac{\theta}{\eta}\right)^{\alpha}}\right)^{n(1-\beta)}$$

By taking the (Ln) to the given function:

$$\text{LnL} \propto n_1\text{Ln}(6\alpha) + n1\alpha Ln(\theta) - (1+\alpha)\sum_{i=1}^{n1} Ln(t_i) + \sum_{i=1}^{n1} Ln\left(e^{-2\left(\frac{\theta}{t_i}\right)^{\alpha}} - e^{-3\left(\frac{\theta}{t_i}\right)^{\alpha}}\right) + nLn(\beta) + (\beta-1)\sum_{i=n1+1}^{n} Ln\left(1 - 3\,e^{-2\left(\frac{\theta}{t_i}\right)^{\alpha}} + 2e^{-3\left(\frac{\theta}{t_i}\right)^{\alpha}}\right)^{\beta-1} + n(1-\beta)Ln\left(1 - 3\,e^{-2\left(\frac{\theta}{\eta}\right)^{\alpha}} + 2e^{-3\left(\frac{\theta}{\eta}\right)^{\alpha}}\right) \quad (22)$$

It is not possible to find the parameters' estimators by the classical methods, so it is possible to use the numerical methods to solve the equations simultaneously.

6. Least squares error method (LSE)

LSE method is a very common method for linear and nonlinear regression models and is used by minimizing the sum of squared errors between the true value and the predicted value.

$$s^*(t) = \sum_{i=1}^{n_1}\left(F_{1(t_{(i)})} - \frac{i}{n+1}\right)^2 + \sum_{i=n_1+1}^{n}\left(F_{2(t_{(i)})} - \frac{i}{n+1}\right)^2$$

By substituting the (CDF) for the (PLAT) of Fréchet distribution given in equation (18), the least squares function can be obtained as follows:

$$s^*(t:\beta,\alpha,\theta) = \sum_{i=1}^{n1}\left[\left(3e^{-2\left(\frac{\theta}{t}\right)^{\alpha}} - 2e^{-3\left(\frac{\theta}{t}\right)^{\alpha}}\right) - \frac{i}{n+1}\right]^2 +$$

$$\sum_{i=n_1+1}^{n}\left[1 - \left[\left(1 - 3\,e^{-2\left(\frac{\theta}{t}\right)^{\alpha}} + 2e^{-3\left(\frac{\theta}{t}\right)^{\alpha}}\right)^{\beta}\left(1 - 3\,e^{-2\left(\frac{\theta}{\eta}\right)^{\alpha}} + 2e^{-3\left(\frac{\theta}{\eta}\right)^{\alpha}}\right)^{1-\beta}\right] - \frac{i}{n+1}\right]^2 \qquad (23)$$

It is not possible to find the parameters' estimators by the classical methods, so it is possible to use the numerical methods to solve the equations simultaneously [14].

7. Maximum product of spacing estimation method (MPS)

the researcher proposed this method to be a strong alternative to the (MLE) to estimate the unknown parameters of continuous univariate distributions. This method is characterized by several properties, including: (stability, sufficiency, consistency). The purpose of this method is to find the parameter estimates for the TMR distribution system [7]. We have a random sample arranged (X1, X2,…., Xn-1) and spaced regularly drawn from a community that follows the TMR distribution system, as it has a (PDF), and a (CDF), and the maximum product of successive differences is obtained by maximizing the geometric mean of the distances [15].

$$D_i(\alpha,\beta,\theta) = F^*_{(t_i)} - F^*_{(t_{i-1})} \qquad i = 1, 2, \ldots\ldots, n+1 \qquad (24)$$

$$G_{(\alpha,\beta,\theta)} = \left[\left(\prod_{i=1}^{n_1}\left[F_{1_{t(i)}} - F_{1_{t(i-1)}}\right]\right)\left(F_{2_{t(n1+1)}} - F_{1_{t(n1)}}\right)\left(\prod_{i=n_1+2}^{n+1}\left[F_{2_{t(i)}} - F_{2_{t(i-1)}}\right]\right)\right]^{\frac{1}{n+1}} \qquad (25)$$

By substituting the (CDF) of the progressive pressure stage (PLAT) for the Fréchet distribution given in Equation (18) in Equation (25), it is possible to obtain the maximum result of successive differences as follows:

$$G_{(\alpha,\beta,\theta)} = \left[\left(\prod_{i=1}^{n_1}\left[\left(3e^{-2\left(\frac{\theta}{t_i}\right)^{\alpha}} - 2e^{-3\left(\frac{\theta}{t_i}\right)^{\alpha}}\right) - \left(3e^{-2\left(\frac{\theta}{t_{i-1}}\right)^{\alpha}} - 2e^{-3\left(\frac{\theta}{t_{i-1}}\right)^{\alpha}}\right)\right]\right) \qquad \left(\left[1 - \left(1 - \right.\right.\right.$$

$$\left.3e^{-2\left(\frac{\theta}{t_{(n_1+1)}}\right)^{\alpha}} + 2e^{-3\left(\frac{\theta}{t_{(n_1+1)}}\right)^{\alpha}}\right)^{\beta}\left(1 - 3e^{-2\left(\frac{\theta}{\eta}\right)^{\alpha}} + 2e^{-3\left(\frac{\theta}{\eta}\right)^{\alpha}}\right)^{1-\beta}\Bigg] -$$

$$\left[3e^{-2\left(\frac{\theta}{t_{(n1)}}\right)^{\alpha}} - 2e^{-3\left(\frac{\theta}{t_{(n1)}}\right)^{\alpha}}\right]\Bigg)\Bigg(\prod_{i=n1+2}^{n+1}\Bigg[1-\left(1-3e^{-2\left(\frac{\theta}{t_i}\right)^{\alpha}}+2e^{-3\left(\frac{\theta}{t_i}\right)^{\alpha}}\right)^{\beta}\left(1-3e^{-2\left(\frac{\theta}{\eta}\right)^{\alpha}}+2e^{-3\left(\frac{\theta}{\eta}\right)^{\alpha}}\right)^{1-\beta}\Bigg]-\Bigg[1-\left(1-3e^{-2\left(\frac{\theta}{t_{i-1}}\right)^{\alpha}}+2e^{-3\left(\frac{\theta}{t_{i-1}}\right)^{\alpha}}\right)^{\beta}\left(1-3e^{-2\left(\frac{\theta}{\eta}\right)^{\alpha}}+2e^{-3\left(\frac{\theta}{\eta}\right)^{\alpha}}\right)^{1-\beta}\Bigg]\Bigg)\Bigg]^{\frac{1}{n+1}} \quad (26)$$

It is not possible to find the parameters' estimators by the classical methods, so it is possible to use the numerical methods to solve the equations simultaneously.

8. Simulation

To evaluate the performance of the maximum likelihood method, the least squares method, and the maximum outcome of successive differences method, a simulation method was conducted, taking five simulation experiments for each of the distribution parameters and for five different sample sizes, n=(10, 25, 50, 75, 100), the default values of the parameter (α=0.9, 0.5, 0.5, 2, 2.5), the default values of the second parameter (β=1.5, 1.5, 1.5, 2, 3.5), and the default values of the third parameter (θ=0.5, 0.9, 0.5, 2.5, 1.5), and to know the efficiency of the estimation methods, using the statistical criterion of the supplementary error squares. The results of the maximum likelihood method were presented in Table (1). From observing the table, it was clear that the estimated values of the parameters and for all experiments were very close to the default values of the system parameters, especially when the sample size increased, which means that the maximum likelihood method was a good estimate of the parameters and can be relied upon.

Table (1): The predictive values to the parameters by using MSE for the reliability function by the (MLE)

n	PARAMETAR			ESTMATOR			$MSE(\hat{R}_{(t)})$
	α	θ	β	α	θ	β	
10	0.9	0.5	0.5	1.075727	0.586208	0.779836	0.003878
25				1.056476	0.568238	0.685592	5.78E-05
50				1.023546	0.550018	0.613918	1.69E-06
75				0.996632	0.536738	0.58103	2.5E-07
100				0.97694	0.528906	0.562545	6E-08
10	0.5	0.9	0.5	0.537889	1.002528	0.578285	0.001164
25				0.535792	0.977985	0.556599	6.66E-06
50				0.527794	0.963295	0.544224	2E-07
75				0.52296	0.938656	0.526185	3E-08
100				0.51852	0.932383	0.520793	1E-08

10	0.5	0.5	0.5	0.619116	0.712806	0.76511	0.001696
25				0.62047	0.653617	0.687065	2.12E-05
50				0.594035	0.605955	0.622271	5.8E-07
75				0.576628	0.586585	0.596741	9E-08
100				0.563544	0.556645	0.561683	2E-08
10	2	2.5	1.5	2.08862	2.528781	2.152875	0.004535
25				2.076425	2.523701	2.062866	3.02E-05
50				2.065873	2.510197	2.006645	7E-07
75				2.052888	2.506024	1.995589	1E-07
100				2.044311	2.502242	1.9877	0.000001
10	2.5	1.5	3.5	2.52963	1.50108	3.637564	0.0017724
25				2.54038	1.501034	3.521185	1.35E-05
50				2.539296	1.498706	3.470984	1.7E-07
75				2.534158	1.498051	3.460654	2E-08
100				2.530431	1.49733	3.454657	1E-08

The results obtained using the least squares method are presented in Table (2). The estimated parameter values for all experiments were found to be very close to the true parameter values, particularly as the sample size increased. This indicates that the least squares method provides accurate parameter estimates and can therefore be considered a reliable estimation approach.

Table (2): The predictive values to the parameters by using MSE for the reliability function by the (LSE)

n	PARAMETAR			ESTMATOR			$\mathbf{MSE}(\hat{\mathbf{R}}_{(t)})$
	α	θ	β	α	θ	β	
10	0.9	0.5	1.5	1.075727	0.586208	1.779836	0.002539
25				1.056476	0.568238	1.685592	0.000121
50				1.023546	0.550018	1.613918	1.77E-05
75				0.996632	0.536738	1.58103	2.63E-06
100				0.97694	0.528906	1.562545	8.90E-07
10	0.5	0.9	1.5	0.537889	1.002528	1.578285	0.000674
25				0.535792	0.977985	1.556599	5.20E-05
50				0.527794	0.963295	1.544224	3.56E-06
75				0.52296	0.938656	1.526185	9.50E-07
100				0.51852	0.932383	1.520793	1.70E-07
10	0.5	0.5	1.5	0.619116	0.712806	1.76511	0.001696
25				0.62047	0.653617	1.687065	0.001121

50				0.594035	0.605955	1.622271	2.50E-05
75				0.576628	0.586585	1.596741	3.77E-06
100				0.563544	0.556645	1.561683	1.05E-06
10	2	2.5	1.5	2.08862	2.528781	2.152875	0.002696
25				2.076425	2.523701	2.062866	0.001121
50				2.065873	2.510197	2.006645	2.50E-05
75				2.052888	2.506024	1.995589	4.99E-06
100				2.044311	2.502242	1.9877	1.03E-06
10	2.5	1.5	3.5	2.52963	1.50108	3.637564	0.007513
25				2.54038	1.501034	3.521185	0.001626
50				2.539296	1.498706	3.470984	2.79E-05
75				2.534158	1.498051	3.460654	4.04E-06
100				2.530431	1.49733	3.454657	1.95E-06

The results of the maximum product of successive differences method are presented in Table (3). The estimated parameter values across all experiments were observed to be very close to the true parameter values, particularly as the sample size increased. This suggests that the method provides accurate parameter estimates and can therefore be considered reliable.

Table (3): The predictive values to the parameters by using MSE for the reliability function by the (MPS)

n	PARAMETAR			ESTMATOR			$MSE(\hat{R}_{(t)})$
	α	θ	β	α	θ	β	
10	0.9	0.5	1.5	0.630071	0.4614377	0.9520647	0.006837
25				0.699901	0.4665926	0.7001989	0.001095
50				0.7083851	0.4705394	0.6523395	0.000277
75				0.7472781	0.4791882	0.6120233	6.34E-05
100				0.7775333	0.4877646	0.5909493	1.63E-05
10	0.5	0.9	1.5	0.4115615	0.876722	0.6878638	0.002425
25				0.4500008	0.8713241	0.5613972	0.000177
50				0.4713967	0.885569	0.5332863	9.92E-06
75				0.4801326	0.8934026	0.5238828	1.40E-06
100				0.4845667	0.8960454	0.5188647	3.40E-07
10	0.5	0.5	1.5	0.3875488	0.5046905	0.7981568	0.00331
25				0.4257417	0.4904119	0.6016421	0.000386
50				0.4538148	0.4914853	0.5573806	3.48E-05
75				0.4681732	0.4943459	0.5401624	4.99E-06

100				0.4748655	0.4963423	0.5305237	1.05E-06
10	2	2.5	1.5	1.751016	2.482986	2.230986	1.73E-04
25				1.880577	2.504826	1.977888	6.00E-06
50				1.938602	2.505306	1.964124	1.00E-06
75				1.956669	2.505261	1.975973	1.00E-06
100				1.967437	2.50414	1.979323	1.00E-06
10	2.5	1.5	3.5	2.167825	1.499394	3.622587	1.47E-03
25				2.352852	1.50492	3.40575	3.50E-04
50				2.427456	1.504087	3.432848	6.50E-05
75				2.454567	1.502912	3.454957	4.90E-06
100				2.465835	1.502274	3.465171	8.90E-07

The estimated parameter values in all experiments were found to be very close to the true values, confirming the accuracy of the experimental results. Furthermore, when the assumed parameter values of the five models were altered, the estimation methods responded accordingly. The parameter estimates did not remain constant; instead, they varied from one model to another, reflecting the influence of changes in the underlying model assumptions. The results showed that all the estimation methods used in estimating the parameters (TMR system distribution) were very close, with a slight difference in the mean square error of the estimation methods used, but in first place the maximum likelihood method was the best in estimating the parameters, in second place was the (LSE) method, and in third place was the (MPS) method.

9. Conclusions:

Three methods were compared to estimate the TMR system under graded stress. The aim of the study is to estimate the distribution parameters of the (TMR) system. Three estimation methods were taken, including (MPS, ML, and LS). In addition, Monte Carlo simulation method was used to analyze the efficiency of the estimation methods. The results showed that the maximum likelihood method was superior to other methods because it obtained the lowest mean square error, especially for large sample sizes. As for small samples, the least squares method showed the lowest mean square error.

10. future work:

As a future works, it is better to apply the (TMR) system analysis under the gradually accelerating life choices in the aspects of health, agriculture, technology, and various industrial aspects by using other estimation methods to estimate the system distribution parameters (Bayesian method, weighted least squares method and so on) and compare them with the estimation methods in this research.

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