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Perturbation Iteration Algorithm for Analytical Investigation of Jeffery–Hamel Flow

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Abstract

The goal of this paper is to present how to calculate an approximate semi-analytical approach to magneto-hydrodynamic (MHD) Jeffery–Hamel flow using a unique form of the perturbation iteration algorithm (PIA) . At one and both grid points, the approximate solution of this issue is determined in the form of a quickly converging series. For various values of Hartmann and Reynolds numbers, results for velocity profiles in divergent and convergent channels are shown. (P IA) and fourth-order Runge Kutta are used to solve the problem (RK4). Both solutions are compared to the variance of the parameter technique (MADM) and (DTM) to see if the method is effective (PIA). The effects of the settings are visualized via graphical simulation for both divergent and converging, channels.

Keywords: Jeffery-Hamel flow, Perturbation Iteration Algorithm PIA(1,1), approximate semi-analytical

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خوارزمية تكرار الاضطراب للتحقيق التحليلية لحل تدفق جيفري هامل

علي جويعد اسود المالكي [1]

المستخلص

الهدف من هذه الورقة هو تقديم كيفية حساب النهج شبه التحليلي التقريبي لتدفق جيفري هامل المغناطيسي الهيدروديناميكي (MHD) باستخدام شكل فريد من خوارزمية تكرار الاضطراب (PIA). عند إحدى نقطتي الشبكة وكلتيهما، يتم تحديد الحل التقريبي لهذه المشكلة في شكل سلسلة متقاربة بسرعة. بالنسبة للقيم المختلفة لأرقام هارتمان ورينولدز، يتم عرض نتائج ملفات تعريف السرعة في القنوات المتباعدة والمتقاربة. يتم استخدام (P IA) و Runge Kutta من الدرجة الرابعة لحل المشكلة (RK4). تمت مقارنة كلا الحلين بتباين تقنية المعلمة (MADM) و (DTM) لمعرفة ما إذا كانت الطريقة فعالة (PIA). يتم تصور تأثيرات الإعدادات عبر المحاكاة الرسومية لكل من القنوات المتباينة والمتقاربة.

الكلمات المفتاحية: تدفق جيفري هامل، خوارزمية تكرار الاضطراب PIA(1,1)، شبه تحليلية تقريبية

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معلومات البحث

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1- Introduction:

The flow between two inclined plates is one of the most commonly used situations in mechanical engineering applications. Jeffery [1] and Hamel [2] lay the mathematical groundwork for this type of flow. The flow between two planes that meet at an angle, known as Jeffery– Hamel flow, was first studied by Jeffery and Hamel. They worked on incompressible viscous fluid flow via convergent-divergent channels and proposed a comparable Navier–Stokes solution in a specific situation of two-dimensional flow in a channel with inclined plane walls meeting at a source or sink vertex. Batchelor [3] contains a plethora of information on Jeffery–Hamel flow. Theoretical research into magneto hydrodynamic channels has found various applications in the design of liquid metal cooling systems, accelerators, pumps, and flow meters [4]. The magnetic field works as a control parameter in Jeffery – Hamel flow, and there are other non-dimensional parameters such as the magnetic Reynolds number and the Hartmann number, in addition to the angle of the walls addressed in this problem. In comparison to traditional issues, a wide range of solutions may be predicted. Many researchers were able to study it, including Abeer Al-Jassem, Abdul-Sattar Al-Safa in [5-11] and Hardik S. Patel, Ramakanta Meher The study included the study of the effect of Reynolds number and the magnetic field on the fluid flow between two plates, and there and other researchers all seek a perfect match between the numerical solution with the analytical solutions. I was able to apply an algorithm and got an almost perfect match between the analysis solution and the numerical solution, traversed the effect of physical parameters on the fluid flow velocity, and studied the method's convergence using within different spaces to prove the validity of the PIA(1,1). studied the analytical research of Jeffery–Hamel flow with a strong magnetic field with nanoparticles using the PIA(1,1) decomposition approach, and [8] explored the rotating MHD viscous flow and heat transfer between stretched and porous surfaces using the analytical method. The goal of this study is to find an analytical approximation solution to the nonlinear problem that characterizes the incompressible viscous. Flow of Jeffery Hamel Furthermore, we investigate PIA,[9] which is based on an algorithm that is categorized according to the number of terms in the perturbation expansion n1 and the degrees of derivatives in the Taylor expansions n_2. This approach, however, has been dubbed PIA $(n_1, n_2), n_1, n_2$ =1,2,.... PIA(n_1, n_2) has awell-known approach for addressing issues in several branches of science and engineering. In this paper, PIA is used to solve a relatively complicated issue, JHF in diverging and converging channels, in order to discover an analytical approximation solution and to understand the influence of physical factors on these answers. The primary benefit of this strategy is that it minimizes computing labor while retaining a better degree of accuracy. Furthermore, PIA (1,1) is examined in order to provide suitable conditions for the convergence of the approximation series solution given by,PIA.

3- Mathematical Formulation

Consider the constant two-dimensional flow of an incompressible conductive viscous fluid between two rigid plane walls that intersect at an angle 2θ, as seen in Figure (1). It is assumed here that the velocity is completely radial and only depends on r and h. In polar coordinates, the continuity equation and Navies–Stokes equation are [11], As in equations (1)-(11):

$$\frac{\partial u_r(r,\theta)}{\partial r} + \frac{1}{r} u_r(r,\theta) = 0, \tag{1}$$

$$-u_r(r,\theta)\frac{\partial u_r(r,\theta)}{\partial r} - \frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u_r(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u_r(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r(r,\theta)}{\partial \theta^2} - \frac{u_r(r,\theta)}{r^2}\right] - \frac{\sigma B_\circ^2}{\rho\, r^2} u_r(r,\theta) = 0, \tag{2}$$

$$-\frac{1}{\rho r}\frac{\partial P}{\partial \theta} + \frac{2\,v}{r^2}\cdot\frac{\partial u_r(r,\theta)}{\partial \theta} = 0, \tag{3}$$

where $B_\circ$ is the electromagnetic induction, r is the fluid conductivity, u_r; h is the radial velocity, p is the fluid pressure, $\nu = \frac{\mu_B}{\rho}$ is the coefficient of kinematic viscosity, and ρ is the fluid density.

Eq. (1) may be expressed as follows:

$$\mathbf{u_r = U_r}, \quad \frac{\partial \mathbf{u_r}}{\partial \boldsymbol{\theta}} = \mathbf{0} \quad \mathbf{at} \quad \boldsymbol{\theta} = \mathbf{0},$$

$$u_r = 0, \quad at \quad \theta = \varphi \qquad (4)$$

Therefore, we will start by deriving the equation using non-dimensional transformations to convert the equations into a simple equation for which an analytic solution can be found [9].

$$-\frac{\partial^2 u_r}{\partial\theta\partial r} - u_r\frac{\partial^2 u_r}{\partial\theta\partial r} - \frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\partial r} + v\left[\frac{\partial^3 u_r}{\partial\theta\partial r^2} + \frac{1}{r}\frac{\partial^2 u_r}{\partial\theta\partial r} + \frac{1}{r^2}\frac{\partial^3 u_r}{\partial\theta^3} - \frac{1}{r^2}\frac{\partial u_r}{\partial\theta}\right] - \frac{\sigma B_\circ^2}{\rho\, r^2}\cdot\frac{\partial u_r}{\partial\theta} = 0, \qquad (5)$$

$$-\frac{1}{\rho}\frac{\partial^2 P}{\partial\theta\partial r} + \frac{1}{\rho r}\frac{\partial P}{\partial\theta} + v\left[\frac{2}{r}\frac{\partial^2 u_r}{\partial\theta\partial r} - \frac{4}{r^2}\frac{\partial u_r}{\partial\theta}\right] = 0, \qquad (6)$$

Now by subtracting the two equations (5) from (6) we get

$$-\frac{\partial^2 u_r}{\partial\theta\partial r} - u_{\dot r}\frac{\partial^2 u_r}{\partial\theta\partial r} - \frac{1}{\rho\dot r}\frac{\partial P}{\partial\dot\theta} + v\left[\frac{\partial^3 u_r}{\partial\theta\partial r^2} - \frac{1}{r}\frac{\partial^2 u_r}{\partial\theta\partial r} + \frac{1}{r^2}\frac{\partial^3 u_r}{\partial\theta^3} + \frac{3}{r^2}\frac{\partial u_r}{\partial\theta}\right] - \frac{\sigma B_\circ^2}{\rho\, r^2}\cdot\frac{\partial u_r}{\partial\theta} = 0, \qquad (7)$$

by integrating both sides with respect to r for continuity equation (1) in the form

$$f(\dot\theta) = r u_r(r,\theta), \qquad (8)$$

the dimensionless variables with the Equation (4), can make the problem dimensionless,

$$E(\eta) = \frac{f(\theta)}{rU_r}, \qquad \gamma = rU_r, \qquad \eta = \frac{\theta}{\varphi}, \qquad u_r(r,\theta) = \frac{\zeta}{r}E(\eta), \qquad (6)$$

From Equation (8), we find the required derivatives

$$\frac{\partial u_r}{\partial\theta} = \frac{\zeta}{r\varphi}\frac{dE(\eta)}{d\eta}, \qquad \frac{\partial^3 u_r}{\partial\theta^3} = \frac{\zeta}{\varphi^3 r}\frac{d^3E(\eta)}{d\eta^3}, \qquad \frac{\partial^3 u_r}{\partial r^2\partial\theta} = \frac{2\zeta}{\varphi r^3}\frac{dE(\eta)}{d\eta},$$

$$\frac{\partial u_r}{\partial r} = \frac{-\zeta}{r^2}E(\eta), \qquad \frac{\partial^2 u_r}{\partial r\partial\theta} = \frac{-\zeta}{\varphi r^2}\frac{dE(\eta)}{d\eta}, \qquad \frac{\partial^2 u_r}{\partial r^2} = \frac{2\zeta}{r^3}E(\eta), \qquad (8)$$

Now, from equation (7) and substituting the above derivative into the Equation (7) yield.

$$-\frac{2\zeta^2}{\varphi r^3}E(\eta)\frac{dE(\eta)}{d\eta} - v\left[\frac{\zeta}{\varphi^3 r^3}\frac{d^3E(\eta)}{d\eta^3} + \frac{4\zeta}{\varphi r^3}\frac{dE(\eta)}{d\eta}\right] - \frac{\sigma B_\circ^2}{\rho\, r^2}\cdot\frac{\zeta}{r\varphi}\frac{dE(\eta)}{d\eta} = 0, \qquad (9)$$

We multiply both sides of the equation by $\frac{\varphi^2}{v} \neq 0$ with the use of the two hypotheses $R_e = \frac{\zeta\varphi}{v}$, $H_a = \sqrt{\frac{\sigma B_\circ^2}{\rho\, v}}$ and by simplifying them, we get the following form

$$\frac{d^3E(\eta)}{d\eta^3} + 2\varphi R_e E(\eta)\frac{dE(\eta)}{d\eta} + (4 - H_a)\varphi^2 \frac{dE(\eta)}{d\eta} = 0, \quad (10)$$

The boundary condition of the problem dimensionless are

$$E(0) = 1, \quad \frac{dE(0)}{d\eta} = 0, \quad E(1) = 0, \quad (11)$$

can classified to two cases as:

- Divergent Channel : $E > 0$, $U_r > 0$,
- Convergent Channel : $E < 0$, $U_r < 0$,

The values of the skin friction coefficient can be obtained by using, as seen in Figure (1).

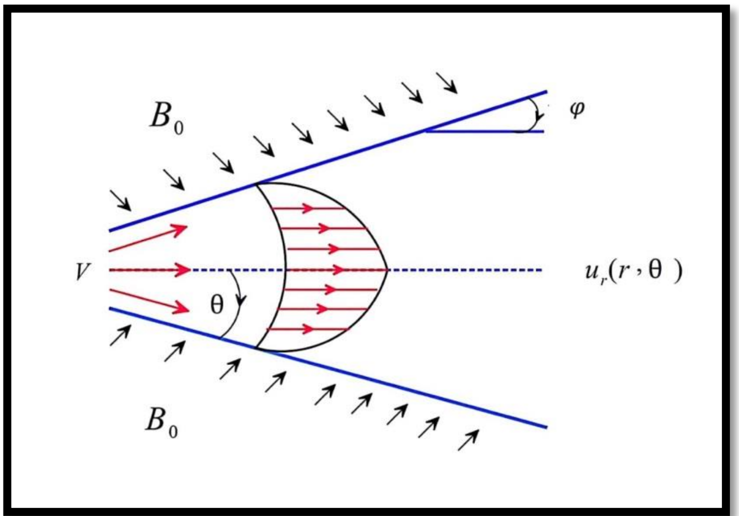

Figure (1): The geometry of the MHD Jeffery – Hamel flow

4-Application of the Jeffery-Hamel Flow Problem

The application of the PIA(1,1) by the steps of its algorithm to the nonlinear differential Equation in order to find an analytical approximate solution,[10] can be illustrated as follows[9], As in equations (12)-(23);

$$\frac{d^3E(\eta)}{d\eta^3} + \varepsilon\left[2\varphi R_e E(\eta)\frac{dE(\eta)}{d\eta} + (4 - H_a)\varphi^2 \frac{dE(\eta)}{d\eta}\right] = 0, \quad (12)$$

subject boundary condition are:

$$E(0) = 1, \quad \frac{dE(0)}{d\eta} = 0, \quad E(1) = 0, \quad (13)$$

the problem of the auxiliary perturbation parameter ε are

$$D\left(E(\eta), \frac{dE(\eta)}{d\eta}, \frac{d^3E(\eta)}{d\eta^3}, \varepsilon\right) = \frac{d^3E(\eta)}{d\eta^3} + \varepsilon\left[2\varphi R_e E(\eta)\frac{dE(\eta)}{d\eta} + (4 - H_a)\varphi^2 \frac{dE(\eta)}{d\eta}\right], \quad (14)$$

perturbation expansions with only one corrections term are given as follows:

$$E_{n+1} = E_n + \varepsilon(E_C)_n, \quad (15)$$

substituting Equation (14) into Equation (15), Taylor series with first order derivative terms about $\mathfrak{z} = 0$, yields

$$\mathbf{D(E_n, E'_n, , E''_n, E'''_n, 0) + \varepsilon[\, D_{E_n}(E_C)_n + D_{E'_n}\,(E_C)'_n \; + \; D_{E'''_n}(E_C)'''_n + \; D_\varepsilon] \; = 0,} \quad (16)$$

Now, we apply Equation (14) to Equation (16) and we get the following derivatives:

$$\mathbf{D_{E_n} = 2\varepsilon\varphi R_e E'_n,}$$

$$\mathbf{D_{E'_n} = 2\varepsilon\varphi R_e E_n + (4 - H_{\mathit{a}})\varphi^2,}$$

$$\mathbf{D_{E'''_n} = 1,} \quad (17)$$

$$\mathbf{D(E_n, E'_n, E'''_n, 0) = E'''_n,}$$

$$\mathbf{D_\varepsilon = 2\varphi R_e E_n E'_n + (4 - H_{\mathit{a}})\kappa^2 E'_n\,,}$$

By calculating all derivatives at $\varepsilon = 0$ and substituting the results into (17) yields the following linear ordinary differential equations

$$\mathbf{(E_c)'''_n = -\frac{1}{\varepsilon} E'''_n\,(\eta) - 2\varphi R_e E_n{}'(\eta) E_n(\eta) - (4 - \mathit{H_a})\kappa^2 E_n{}'(\eta),} \quad (18)$$

assume that the initial condition [14] ,

$$\mathbf{E_o(\eta) = \lambda_1 + \lambda_2\eta + \lambda_3 \frac{\eta^2}{2!}\,,} \quad (19)$$

Where,

$$\mathbf{E(0) = \lambda_1,\; E'(0) = \lambda_2,\;\; E''(0) = \lambda_3\,,} \quad (20)$$

from the boundary condition(20), we get

$$\mathbf{E_o = 1 + \lambda_3 \frac{\eta^2}{2},} \quad (21)$$

Now, we have obtained a preliminary condition for solving the problem that contains λ_3 is unknown. We can obtain the value of λ_3 from the analytical approximate solution of Equation (18) at $\eta = 1$. We get the analytical approximate solutions in the following equations:

$$\mathbf{E_1 = 1 + \frac{1}{2}\lambda_3\, \eta^2 - \frac{1}{4}\left(\frac{1}{3}\, \lambda_3 R_e \varphi + \frac{1}{6}(4 - \mathit{H_a})\lambda_3 \varphi^2\right)\eta^4 - \frac{1}{120}{\lambda_3}^3 R_e \varphi\, \eta^6\,,} \quad (22)$$

$$\mathbf{E_2 = 1 + \frac{1}{2}\lambda_3\, \eta^2 - \frac{1}{4}\left(\frac{1}{3}\, \lambda_3 R_e \varphi + \frac{1}{6}(4 - \mathit{H_a})\lambda_3 \varphi^2\right)\eta^4 - \frac{1}{120}\lambda_3^2 R_e \varphi\, \eta^6 - \frac{1}{6}\,(\frac{1}{10}\, R_e \varphi}$$

$$\mathbf{\left(-\frac{1}{3}\lambda_3 R_e \varphi\; - \frac{1}{6}(4 - \mathit{H_a})\, \varphi^2\, \lambda_3\right) + \frac{1}{20}\,(4 - \mathit{H_a})\varphi^2(-\frac{1}{3}\, \lambda_3\, R_e\, \varphi - \frac{1}{6}(4 - \mathit{H_a})}$$

$$\mathbf{\varphi^2\, \lambda_3)\,)\, \eta^6 + \frac{1}{21}\, R_e\, \varphi\, (-\frac{1}{12}\lambda_3\, R_e\, \varphi - \frac{1}{24}(4 - \mathit{H_a})\varphi^2 \lambda_3)\lambda_3 - \frac{1}{840}(4 - \mathit{H_a})\lambda_3^2}$$

$$\mathbf{R_e\varphi^3)\eta^6 - \frac{1}{10}(-\frac{1}{1080}\lambda_3^2\, R_e^2\, \varphi^3 + \frac{1}{36}R_e\varphi\,(-\frac{1}{12}\,\lambda_3 R_e\varphi - \frac{1}{24}\,(4 - H_a)\varphi^2\,\lambda_3)}$$

$$\mathbf{(-\frac{1}{3}\lambda_3\, R_e\,\varphi - \frac{1}{6}(4 - H_a)\,\varphi^2\;\lambda_3\,)\,)\,\eta^{10} - \frac{1}{12}(-\frac{1}{1100}\,\lambda_3^2\,\varphi^2(-\frac{1}{12}\,\lambda_3\,R_e\varphi - \frac{1}{24}}$$

$$\mathbf{(4 - H_a)\,\varphi^2\,\lambda_3)\lambda_3^2 - \frac{1}{6600}\lambda_3^2 R_e^2\varphi^3(-\frac{1}{3}\lambda_3 R_e\varphi - \frac{1}{6}(4 - H_a)\varphi^2\lambda_3))\eta^{12} - \frac{1}{2620800}}$$

$$\mathbf{\lambda_3^4\, R_e^3\,\varphi^3\;\;\eta^{14}\;\; - \frac{1}{8}(-\frac{1}{240}\,\lambda_3^2\, R_e^2\varphi^2 + \frac{1}{42}\lambda_3 R\,_e\varphi(-\frac{1}{3}\,\lambda_3\, R_e\varphi - \frac{1}{6}(4 - H_a)\varphi^2\lambda_3\,,} \qquad \mathbf{(23)}$$

5- Convergence and Error Estimate of PIA

In this section,[13-15] the description of convergence of approximate analytical solution that results from applying the perturbation iteration algorithm to the nonlinear system equations is discussed. In addition to, the derivation of the condition of convergence to test the approximate analytical solutions is determined. That is, we study the convergence of the approximate analytical solution by giving some definitions and theorems as follows:

Definition 5.1.: [13] *that B is Banach space,* R *is the real numbers and* Q[E] *is a nonlinear operator defined as* $\mathbb{G}[\mathrm{E}]: B \longrightarrow \mathbb{R}$ *Then the sequence of generated solutions by perturbation iteration algorithm can be written as*:

$$\mathbb{G}[\mathbf{E}_m] = \mathbf{E}_{m-1} + (\mathbf{E}_c)_{\,m-1}\,, \quad m = 1, 2, \ldots, \quad \mathbf{(24)}$$

Definition 5.2. :[14] *Let* Q[Ψ] *satisfies Lipchitz condition such that for*, $0 < \xi < 1, \xi \epsilon \mathrm{R}$, *we* get

$$\|\mathbb{G}[\mathbf{E}_m] - \mathbb{G}[\mathbf{E}_{m-1}]\;\;\| \le \xi \|\mathbf{E}_m - \mathbf{E}_{m-1}\| \qquad \mathbf{(25)}$$

Definition 5.3. :[14] let $p \le 1$ be a real number . The $p - norm$ and $\infty - norm$ of the vector's $X = (\,x_1, x_2, \ldots, x_n)^T$ *is*

$$\|\mathbf{X}\|_{\mathbf{p}} = (\textstyle\sum_{\mathbf{i=1}}^{\mathbf{n}} |\mathbf{x_i}|^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}, \quad \|\mathbf{X}\|_\infty = \mathbf{Max_i}\,|\mathbf{x_i}| \;\; . \;\; \mathbf{(26)}$$

Theorem 5.1 : [13] Suppose that B be a Banach space , Q : B → B is a nonlinear mapping and $\|\mathbb{G}\,[\mathrm{E}_m] - \mathbb{G}\,[\mathrm{E}_{m-1}]\| \le \xi \|\mathrm{E}_m - \mathrm{E}_{m-1}\|$, $\Psi \in \mathrm{Q}$ for some constant $0 < \zeta < 1$, then Q has a unique fixed point. The sequence $\mathbb{G}\,[\mathrm{E_m}] = \mathrm{E}_{m+1}$ with an arbitrary choice of $\Psi_0 \in Q$, converges to the fixed point of $\mathbb{G}$ and $\|\mathbb{G}[\mathrm{E}_m] - \mathbb{G}[\mathrm{E}_{m-1}]\;\;\| \le \xi^{m-1}\|\mathrm{E}_1 - \mathrm{E}_0\|$.

Theorem 5.2 : [15] the solution series $Q_m = \{\mathrm{E}_m\}_{m=0}^{\infty}$ convergent if there exist ζ such that ,

$\|C_i\| \le \zeta_i\, \|C_{i-1}\|\,, i = 1,2, \ldots$

Proof : let a sequence as :

$\mathbb{G}_0 = C_0$,

$\mathbb{G}_0 = \mathbb{G}_0 + (\mathbb{G}_C)_0 = C_0 + C_1$,

$\mathbb{G}_1 = \mathbb{G}_1 + (\mathbb{G}_C)_1 = C_0 + C_1 + C_2$,

$$\mathbb{G}_m = \mathbb{G}_{m-1} + (\mathbb{G}_C)_{m-1} = \boldsymbol{C_0} + \boldsymbol{C_1} + \boldsymbol{C_2} + \cdots + \boldsymbol{C_m} = \textstyle\sum_{i=0}^{m} \boldsymbol{C_i}\,, \qquad \mathbf{(27)}$$

We need to show that $\{\mathbb{G}_m\}_{m=0}^{\infty}$ is a Cauchy sequence in B consider that

$$\|\mathbb{G}_{m-1} - \mathbb{G}_m\| = \|C_{m+1}\| \leq \xi \|C_m\| \leq \xi^2 \|C_{m-1}\| \leq \cdots \leq \xi^{m+1} \|C_0\| ,$$

For $r, m \in N$, $r \geq m$ we have

$$\|\mathbb{G}_r - \mathbb{G}_m\| = \|\mathbb{G}_r - \mathbb{G}_{r-1} + \mathbb{G}_{r-1} + \mathbb{G}_{r-2} - \mathbb{G}_{r-2} + \cdots + \mathbb{G}_{m+1} + \mathbb{G}_m\|,$$

$$\leq \|\mathbb{G}_r - \mathbb{G}_{r-1}\| + \|\mathbb{G}_{r-1} - \mathbb{G}_{r-2}\| + \cdots + \|\mathbb{G}_{m+1} + \mathbb{G}_m\|,$$

$$\leq \xi^r \|C_0\| + \xi^{r-1} \|C_0\| + \cdots + \xi^{m+1} \|C_0\|,$$

$$\leq \frac{1 - \xi^{r-m}}{1 - \xi} \xi^{m+1} \|C_0\|$$

Since $0 < \xi < 1$, we can obtain $\lim_{r,m\to\infty} \|\mathbb{G}_r - \mathbb{G}_m\|$. Thus, $\{\mathbb{G}_m\}_{m=0}^{\infty}$ is a Cauchy sequence in B and this implies that the solution series is convergent.

Theorem 5.3 : [16] Assume that the solution series $\sum_{i=0}^{\infty} \mathrm{C_i}$ is convergent to the solution Q[E] . If the truncated series $\sum_{i=0}^{m} \mathrm{C_i}$ is an approximation to the solution Q[E] of the non-linear system equations, then the maximum error estimate is: [15]

$$T_m \leq \frac{\xi^{m+1}}{1-\xi} \|C_0\|, \qquad (28)$$

Proof. From Theorem (5.2) for $r \geq m$, yield

$$\|\mathbb{G}_r - \mathbb{G}_m\| \leq \frac{1-\xi^{r-m}}{1-\xi} \xi^{m+1} \|C_0\| , \qquad (29)$$

and in addition to

$$Q = Lim_{r\to\infty} Q_r(\eta) = \sum_{i=0}^{\infty} C_i \ , \qquad (30)$$

we can write

$$\|Q - \sum_{i=0}^{\infty} C_i \| \leq \frac{\zeta^{m+1}}{1-\zeta} \|C_0\|, \qquad (31)$$

since $1 - \xi^{r-m} < 1$, Equation (46) can be written as:

$$T_m = \|Q - \sum_{i=0}^{\infty} C_i \| \leq \frac{\xi^{m+1}}{1-\xi} \|C_0\|, \qquad (32)$$

To make a decision the convergence of PIA and from Theorems (39)-(41), we can summarize the convergence condition in the following definition:

Definition 5.4: [16] (The condition of Convergence) for i = 0,1,2, ..., we can define [9]

$$\zeta_i = \begin{cases} \dfrac{\|C_{i+1}\|}{\|C_i\|}, & \|C_i\| \neq 0, \\ 0, & \|C_i\| = 0, \end{cases}$$

Then, We can state that the analytical approximate solution series $\{\mathrm{E}_m\}_{m=0}^{\infty}$ converges to the exact solution when $0 < \xi < 1$, for all $i = 0,1,2,\ldots,$. In general, Because of the difficulty in finding their exact solutions, we can say that the critical part of accepting these solutions is the study of convergence of solutions of systems of equations resulting from Jeffery Hamel flow analysis, and this is encouraged to use the perturbation iteration algorithm that gives series of the approximate analytical solution. As a result, we will investigate the convergence of all solutions using the perturbation iteration approach on the Jeffery Hamel flow.

5.1- Convergence Test for the Solution of JHFCF Problem

The convergence analysis for the approximate analytical solution is introduced as a consequence of using PIA to solve Jeffery Hamel's flow. By computing the values of ζ in the definition (5.4) with $L_1 - norm$, $L_2 - norm$ and $L_\infty - norm$, the converge is achieved. The values of ζ each solution is displayed. in Tables (1)-(4), Look at the Figure from (2)-(9):as the following:[13-1]

Table (1): The values of ζ for $R_e = 50, \ \varphi = 5^{\circ}, H_a = 1000$.

	$L_1 - norm$,	$L_2 - norm$	$L_\infty - norm$
ζ_0	0.0109652739	0.0109652739	0.0109652739
ζ_1	0.0053587384	0.0037892002	0.0026793692
ζ_2	0.0002543935	0.0001468742	0.0000847978
ζ_3	0.0000052733	0.0000026366	0.0000013183
⋮			

Table (2): The values of ζ for $R_e = 50, \ \varphi = -5^{\circ}, H_a = 1000$.

	$L_1 - norm$,	$L_2 - norm$	$L_\infty - norm$
ζ_0	0.4809037000	0.4809037000	0.4809037000
ζ_1	0.2349124260	0.1661081694	0.1174562130
ζ_2	0.0538261737	0.0310765559	0.0179420579
ζ_3	0.0076109034	0.0038054517	0.0019027258
⋮			

Table (3): The values of ζ for $R_e = 80, \ \varphi = 5^{\circ}, H_a = 0.0$.

	$L_1 - norm$,	$L_2 - norm$	$L_\infty - norm$
ζ_0	0.0452585849	0.0452585849	0.0452585849
ζ_1	0.0428970086	0.0303327657	0.0214485043
ζ_2	0.0046186725	0.0026665918	0.0015395575
ζ_3	0.0003131935	0.0001565968	0.0000782984
⋮			

Table (4): The values of ζ for $R_e = 80, \ \varphi = -5^{\circ}, H_a = 0.0$.

	$L_1 - norm$,	$L_2 - norm$	$L_\infty - norm$
ζ_0	0.3451784699	0.3451784699	0.3451784699
ζ_1	0.1132945818	0.0801113671	0.0566472909
ζ_2	0.0179249150	0.0103489545	0.0059749717
ζ_3	0.0017918046	0.0008959023	0.0004479512
⋮			

From Tables (1)-(4), we have, the solution series $E(\eta)$ is convergent.

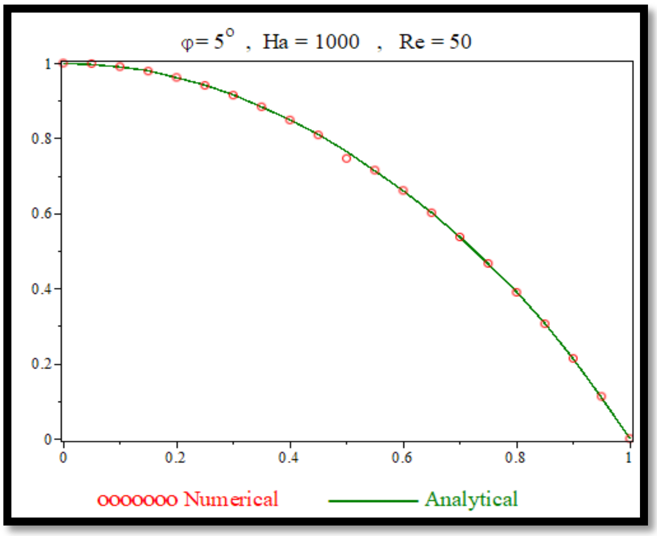

Figure (2): Comparison between Numerical solution and PIA in divergent Channels

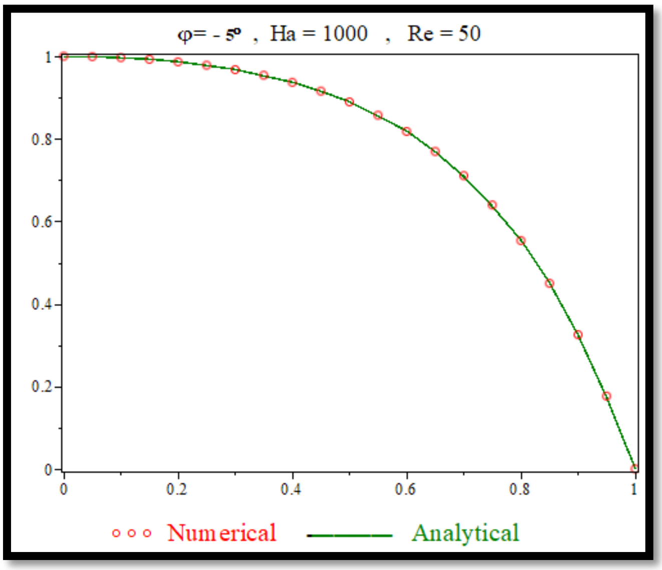

Figure (3): Comparison between Numerical solution and PIA in convergent Channels

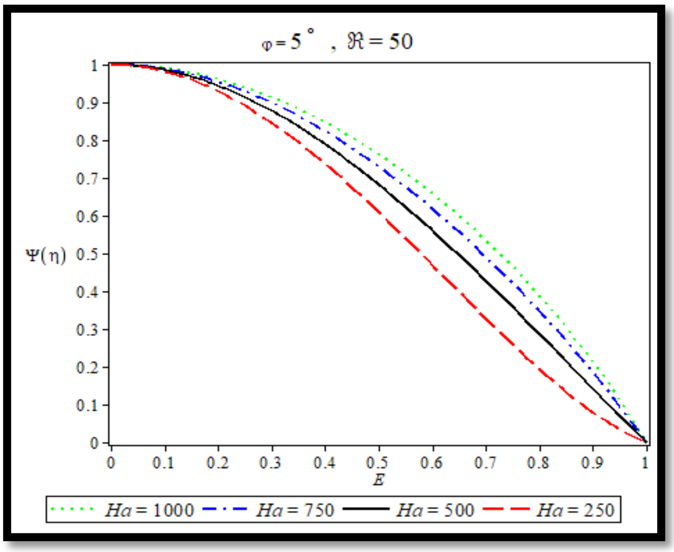

Figure (4): Effect of Hartmann number on velocity profile in divergent channel

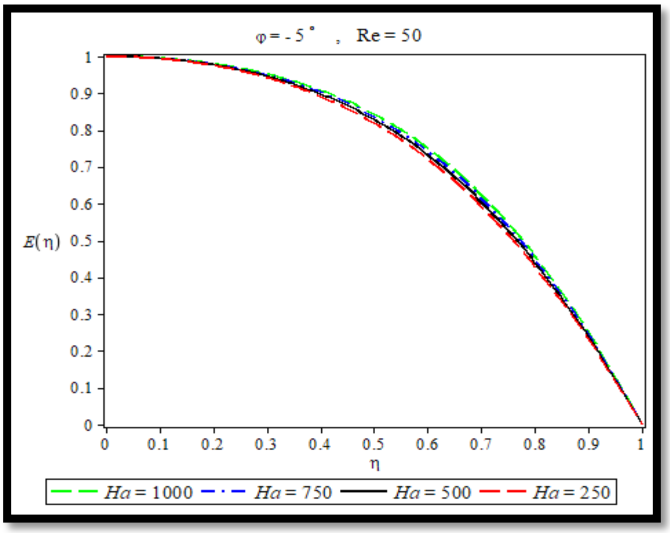

Figure (5): Effect of Hartmann number on velocity profile in convergent channel

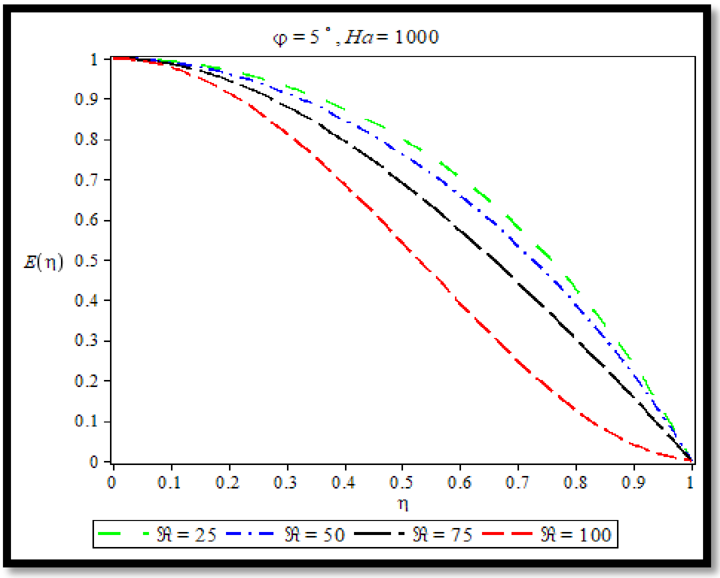

Figure (6): Effect of Reynolds number on velocity profile in divergent channel

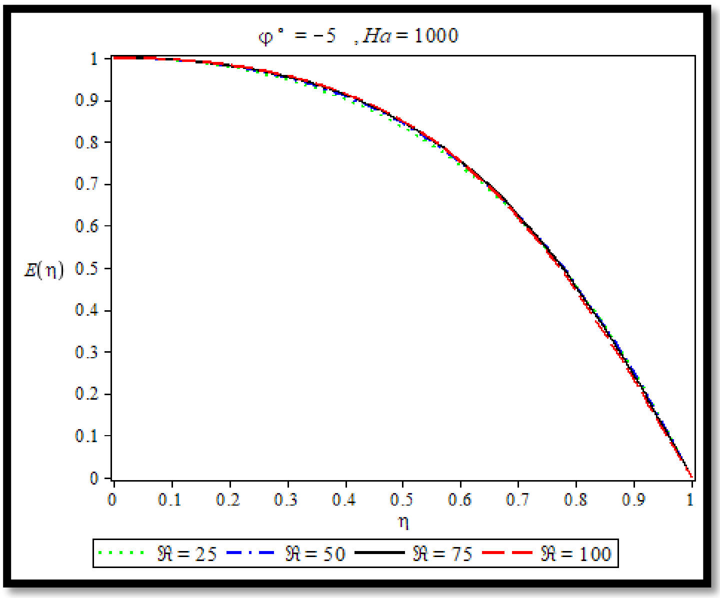

Figure (7): Effect of Reynolds number on velocity profile in convergent channel

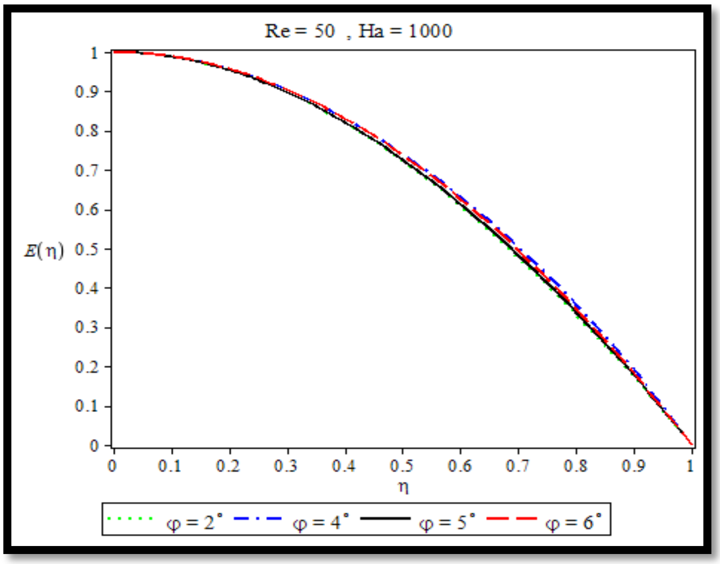

Figure (8): Effect of different angle on velocity profile in divergent channel

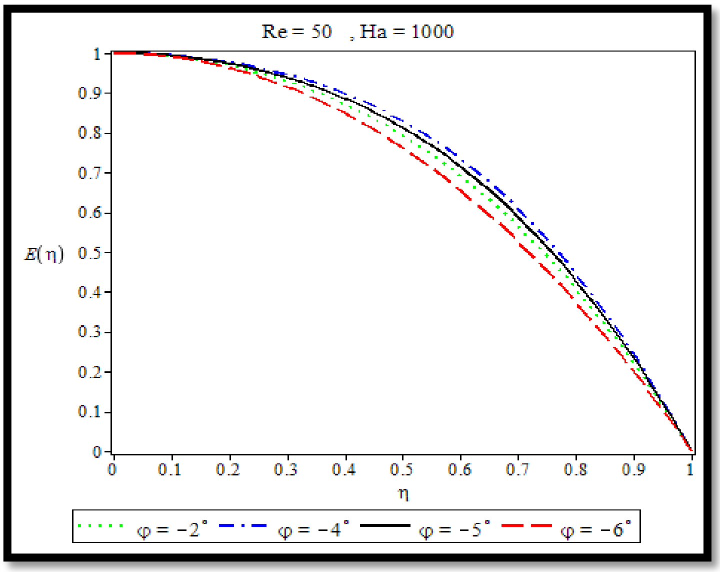

Figure (9): Effect of different angle on velocity profile in convergent channel

5.2: Results and Discussions

In this paper, a few situations of the MHD Jeffery-Hemal flow issue are described, and the resultant approximate solution is compared to known numerical data. The comparison of PIA(1,1) solution with numerical results[12] of Jeffery-Hamel flow for divergent and convergent channels with $R_e = 50$ *and* $H_a = 1000$, is show in Table -8- and Table -9- Following the influence, it is obvious that there is strong agreement between available numerical findings using the PIA(1,1) for Re=50 and Ha=1000 from Figure (2) and Figure (3) for various values of φ . Tables (4) and (5) show the errors. Figure (2) and Figure (3): Compare the numerical and analytical solutions obtained using the Runge –Kutta Method (4) approach. And we discovered an exceptional congruence between the analytical and numerical (RK-4) solutions. We compared the analytical procedures MADM and DTM , as described in the study [10,7] This demonstrates the excellent efficiency of the analytical method employed .The error bar indicates an adequate agreement between the observed data, confirming the PIA (1,1) validity. The mistake is introduced in these tables as follows:

Error $= |E(\eta)PIA - E(\eta)NM|$ Finally it can be concluded that:

- When $\varphi = +5^{\circ}$ and the channel steepness is divergent, the velocity profile falls as the value of Re number increases.
- When $\varphi = -5^{\circ}$ and the channel steepness is convergent, the value of the velocity profile grows with Re number.

5.3: Conclusion

In this research, the magneto hydrodynamics Jeffery–Hamel flow is solved using an analytical approach known as Decomposition Method PIA (1,1) and the results are compared to the numerical results produced by the Runge–Kutta Method of 4th order with its convergence studies to demonstrate the method's efficacy. It is discovered that PIA (1,1) is a powerful strategy for addressing this problem, and it is also discovered that there is a good agreement between the present and numerical results. As a result, increasing Reynolds number with higher angles need a high Hartmann number in order to reduce back flow .see Table (5) - (9)

Table (5): The convergence of the values λ_3

Approximation Order	$,R_e = 50$, $H_a = 1000$		$R_e = 80$, $H_a = 0.0$	
η	$\varphi = 5^o$	$\varphi = -5^o$	$\varphi = 5^o$	$\varphi = -5^o$
1	-1.9159804710	-0.8714791912	-4.9191210920	-0.9771401604
2	-1.9167621820	-0.6819428394	-4.7405802940	-0.8236638726
3	-1.9169515000	-0.6462505891	-4.8500326160	-0.8014452172
4	-1.9169575830	-0.6410301571	-4.8431145610	-0.7989375360
5	-1.9169575860	-0.6404791585	-4.8431145640	-0.7989375443
6	-1.9169575860	-0.6404791585	-4.8431145640	-0.7989375443
7	-1.9169575860	-0.6404791585	-4.8431145640	-0.7989375443
8	-1.9169575860	-0.6404791585	-4.8431145640	-0.7989375443

Table (6): Comparison between PIA(1,1) and Numerical solution [16] in divergent channel

	$R_e = 50$	$H_a = 1000$	$\varphi = 5^{\circ}$
η	PIA(1,1)	Numerical	Error
0.00	1.0000000000	1.0000000000	0.00E+00
0.05	0.9976043725	0.9976051266	1.00E-02
0.10	0.9904242203	0.9904272154	2.99E-06
0.15	0.9784789668	0.9784856266	6.65E-06
0.20	0.9617984382	0.9618100752	1.16E-05

0.25	0.9404190830	0.9404368631	1.77E-05
0.30	0.9143787524	0.9144036551	2.49E-05
0.35	0.8837100957	0.8837428679	3.27E-05
0.40	0.8484326103	0.8484737175	4.11E-05
0.45	0.8085433625	0.8085929634	4.96E-05
0.50	0.7640063377	0.7460642437	1.79E-01
0.55	0.7147403062	0.7148059216	6.56E-05
0.60	0.6606049793	0.6606772549	7.22E-05
0.65	0.6013850857	0.6014624512	7.73E-05
0.70	0.5367718054	0.5368520846	8.02E-05
0.75	0.4663407430	0.4664210525	8.03E-05
0.80	0.3895252749	0.3896018881	7.66E-05
0.85	0.3055836393	0.3056518077	6.81E-05
0.90	0.2135574908	0.2136112049	5.37E-05
0.95	0.1122187177	0.1122503834	3.16E-05
1.00	0.0000000000	0.0000000000	0.00E+00

Table (7): Comparison of PIA(1,1) with MADM[15] , DTM [17] for $\text{Error}_{(1)}$: $|PIA(1,1) - \text{Numerical}|$, $\text{Error}_{(2)}$:$|\text{MADN} - \text{Numerical}|$, $\text{Error}_{(3)}$:$|\text{DTM} - \text{Numerical}|$

$R_e = 80$	$H_a = 0.0$	$\varphi = -5^{\circ}$					
η	PIA(1,1)	Numerical [12]	MADN [7]	DTM [4]	$\text{Error}_{(1)}$	$\text{Error}_{(2)}$	$\text{Error}_{(3)}$
0.0	1.0000000000	1.0000000000	1.0000000000	1.000000000	0.00E+00	0.00E+00	0.00E+00
0.1	0.9959587788	0.9967567004	0.9953253809	0.996121388	7.97E-03	1.43E-03	6.35E-04
0.2	0.9832681744	0.9864914948	0.9806560227	0.983941748	3.22E-02	5.83E-3	2.54E-03
0.3	0.9601635142	0.9675165808	0.9540554254	0.96176774	7.35E-02	1.34E-02	5.74E-02
0.4	0.9234930600	0.9367379265	0.9122960795	0.926570689	1.32E-01	2.44E-02	1.01E-01
0.5	0.8684161151	0.8892083429	0.8508594860	0.873676921	2.07E-01	3.83E-02	1.55E-01
0.6	0.7880338752	0.8174612678	0.7639361363	0.79633424	2.94E-01	5.35E-02	2.11E-01
0.7	0.6730760660	0.7106233991	0.6444255229	0.685154535	3.75E-01	6.61E-02	2.54E-01
0.8	0.5119231950	0.5533874550	0.4839361363	0.527432523	4.14E-01	0.94E-02	2.59E-01
0.9	0.2915101496	0.3251413493	0.2727854661	0.306340629	3.36E-01	5.23E-02	2.88E-01
1.0	0.0000000000	0.0000000000	0. 000000000	0.000000000	0.00E+00	0.00E+00	0.00E+00

Table (8): Comparison between PIA(1,1) and Numerical solution [16] in divergent channel

	$R_e = 50$	$H_a = 1000$	$\varphi = -5^{\circ}$
η	PIA(1,1)	Numerical	Error
0	1.0000000000	1.0000000000	0.00E+00
0.05	0.9991966790	0.9991973819	7.03E-07
0.10	0.9967538845	0.9967567004	2.81E-06
0.15	0.9925718683	0.9925782199	6.35E-06
0.20	0.9864801708	0.9864914948	1.13E-05
0.25	0.9782311875	0.9782489428	1.77E-05
0.30	0.9674909394	0.9675165808	2.56E-05
0.35	0.9538268449	0.9538617983	3.49E-05
0.40	0.9366922558	0.9367379265	4.56E-05
0.45	0.9154075329	0.9154652381	5.77E-05
0.50	0.8891374874	0.8892083429	7.08E-05
0.55	0.8568651603	0.8569499740	8.48E-05
0.60	0.8173621793	0.8176412678	2.79E-03
0.65	0.7691563888	0.7692663193	1.09E-03
0.70	0.7104981879	0.7106233991	1.25E-03
0.75	0.6393281179	0.6394623698	1.34E-03
0.80	0.5532498682	0.5533874550	1.37E-03
0.85	0.4495151448	0.4496468019	1.31E-03
0.90	0.3250298760	0.3251413493	1.11E-03
0.95	0.1763950604	0.1764654178	7.03E-05
1.00	0.0000000000	0.0000000000	0.00E+00

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