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Time Fractional Burgers' Equation

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Abstract

The Burgers' equation is a time-dependent, nonlinear partial differential equation that finds applications in various scientific and technical domains, such as heat conduction, traffic flow, gas dynamics, and non-linear acoustics. This research article provides an overview of Burgers' equations, including their basic form and the time fractional form. The conditions under which exact solutions are available are discussed, and an approximation analytic Adomian approach is introduced to address the standard Burgers' equation.

The paper highlights the challenges in numerically solving the Burgers' equations due to their nonlinear behavior and low viscosity. Analytical solutions require infinite series with slow convergence for low viscosity coefficients. Consequently, numerical approaches are of significant interest for obtaining solutions to the Burgers' equation.

Furthermore, the article discusses how Burgers' equations can be derived from more complex models, making them commonly referred to as toy models. Examples of such simplifications are presented, and the concept of the outer product of vectors is introduced to aid in the understanding of Burgers' equations.

Keywords: Burgers' equation, Analytical solutions, numerical approaches

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معادلة البرجر الكسرية للزمن

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المستخلص

معادلة برجر هي معادلة تفاضلية جزئية غير خطية تعتمد على الوقت، ولها تطبيقات في مجالات علمية وتقنية مختلفة، مثل التوصيل الحراري، وتدفق حركة المرور، وديناميكيات الغاز، والصوتيات غير الخطية. تقدم هذه المقالة البحثية نظرة عامة على معادلات برجر، بما في ذلك صيغتها الأساسية والصيغة الكسرية للزمن. تتم مناقشة الشروط التي تتوفر بموجبها الحلول الدقيقة، ويتم تقديم نهج Adomian التحليلي التقريبي لمعالجة معادلة Burgers القياسية

يسلط البحث الضوء على التحديات التي تواجه حل معادلات برجر عدديًا بسبب سلوكها غير الخطي ولزوجتها المنخفضة. تتطلب الحلول التحليلية سلسلة لا نهائية من التقارب البطيء لمعاملات اللزوجة المنخفضة. وبالتالي، فإن الأساليب العددية ذات أهمية كبيرة للحصول على حلول لمعادلة برجر.

علاوة على ذلك، يناقش المقال كيف يمكن استخلاص معادلات برجر من نماذج أكثر تعقيدًا، مما يجعلها يشار إليها عادةً باسم نماذج الألعاب. يتم عرض أمثلة على هذه التبسيطات، ويتم تقديم مفهوم المنتج الخارجي للمتجهات للمساعدة في فهم معادلات برجر.

الكلمات المفتاحية: معادلة برجر، الحلول التحليلية، الأساليب العددية

انتساب الباحث

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معلومات البحث

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Introduction

Numerous scientific and technical domains, including heat conduction, traffic flow, gas dynamics, and non-linear acoustics, have investigated Burgers' equations in great detail Burgers' equations is a time-dependent, nonlinear partial differential that appears in many forms.

The basic and generic version of Burgers' equation is presented in this section and also we present the time fractional form of Burgers' equation [1,2,3]. We also explain about application of Burgers' equations and the conditions under which the exact solution is available. Also, To answer the standard

Burgers' equation, we formalize the approximation analytical Adomian approach. In the section, we will look at a solution to the time-fractional Burgers' equation.

2. Literature review

2.1. Burgers' equation

Throughout this chapter, assume that $\Omega_T = (0,1) \times (0,T]$. For $(x,t) \in \Omega_T$ as the domain of a given field $w(x,t)$, the general form of Burgers' equation is given by as in equation (1)

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = \epsilon\frac{\partial^2 w}{\partial x^2}, \quad (1)$$

where $\epsilon > 0$ is the singularly perturbation parameter and it is called the diffusion coefficient. The equation (1) is a non linear parabolic partial variance calculation. It signifies the most basic nonlinear dissemination and diffusive impact coupled partial differential equation.

While the right hand side of (1) is omitted (i.e. $\epsilon = 0$), Burgers' equation becomes the inviscid Burgers' equation: as in equation (2)

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = 0, \quad (2)$$

which is a hyperbolic equation. It is important to note that Equation (2) may be rewritten as shown equation (3)

$$\frac{\partial w}{\partial t} + \frac{\partial(\frac{1}{2}w^2)}{\partial x} = 0. \quad (3)$$

A general form of Burgers' equation is suggested by [1] as shown equation (4)

$$\frac{\partial w}{\partial t} + \eta\left(w^q\frac{\partial w}{\partial x}\right) - \epsilon\frac{\partial^2 w}{\partial x^2} - \beta w(x,t)(1 - w(x,t)) = H(x,t) \quad (4)$$

where η and β are parameters, ϵ is the viscosity parameter q, is an integer and H is the source term. Time fraction form of Burgers' equation (4) can be given by [1] as shown equation (5)

$$\frac{\partial^\alpha w}{\partial t^\alpha} + \eta\left(w^q\frac{\partial w}{\partial x}\right) - \epsilon\frac{\partial^2 w}{\partial x^2} - \beta w(x,t)(1 - w(x,t)) = H(x,t) \quad (5)$$

where α is a positive real number.

Equation (5) was first introduced in [2] and in [3] later observed it in 1948, and included it into the turbulence model that year. The name "Burgers' equation" or "Bateman-Burgers equation" has now been commonly used to refer to this kind of equation. Nonlinear behaviour and a low viscosity make numerically solving the Burgers' equations difficult. rainfall forecast models were provided by [4], and they were based on Burgers' equation By converting Eq. (1) to the linear diffusion equation, by [5] and [6] showed that it can be solved analytically for a variety of initial conditions. An infinite series with a very slow convergence is needed for the analytical solution for low values of the viscosity coefficient. To satisfy the need for a solution to the Burgers' equation, numerical approaches are of major interest due to the difficulty of finding analytical solutions.

2.2. Other types of Burgers' equation

The other types of Burgers' equation are as follows: as shown in equation (6)

$$\frac{\partial w}{\partial t} + Cw^2\frac{\partial w}{\partial x} = \epsilon\frac{\partial^2 w}{\partial x^2}, \quad (6)$$

where C is a constant value. It is referred to as cubic Burgers' equation. [7] ,[8]. The modified Burgers' equation that is a semi-linear equation is given by: as shown in equation (7)

$$\frac{\partial w}{\partial t} + w^3\frac{\partial w}{\partial x} = \frac{\epsilon}{2}\frac{\partial^2 w}{\partial x^2}. \quad (7)$$

The equation (7) was derived by. [9]. Another form of Burgers' equation includes the following form: as shown in equation (8)

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = w\frac{\partial^2 w}{\partial x^2}. \quad (8)$$

Lastly, a well-known model including convective effects in fluid diffusion [10] and. [11] reads as shown in equation (9)

$$\frac{\partial w}{\partial t} + w^2\frac{\partial w}{\partial x} = \epsilon w^2\frac{\partial^2 w}{\partial x^2}. \quad (9)$$

2.3. Converting to the Burgers' model

The simplification of some complex and sophisticated models can lead to Burgers' equations. Therefore, it is commonly viewed as a toy model. In simple terms, Burgers' equation is a tool to explain part of the internal behavior of a generic issue. Now, we present two examples. Before that, we need the following definition:

Definition 1. The outer product of two vectors $u \in \mathbb{R}^{n\times 1}$ and $v \in \mathbb{R}^{n\times 1}$ is denoted by $u\otimes v$ and is defined by $\boldsymbol{u}\otimes\boldsymbol{v} = \boldsymbol{u}.\boldsymbol{v}^T$,

where the superscript T shows the transpose of the vector v.

2.4. Navier Stokes equations

Take into account the Navier-Stokes equations as shown equation(10)

$$\begin{cases} \nabla.\mathbf{w} = 0 \\ \dfrac{\partial(\rho\mathbf{w})}{\partial t} + \nabla.(\rho\mathbf{w}\otimes\mathbf{w}) + \nabla P - \mu\nabla^2\mathbf{w} = 0, \end{cases} \quad (10)$$

where ρ represents pressure, **W** represents velocity, and μ represents viscosity. These equations define the motion of a gravitationally insignificant, incompressible, divergence-free (10) flow.

Letting $\boldsymbol{w} = (w_1, w_2)$, simplification in (10) of the x component of the velocity vector gives as shown equation (11)

$$\rho\frac{\partial w_1}{\partial t} + \rho w_1\frac{\partial w_1}{\partial x} + \rho w_2\frac{\partial w_2}{\partial y} + \frac{\partial P}{\partial x} - \mu\left(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2}\right) = 0. \quad (11)$$

This equation simplifies to the following if we assume one dimension problem and neglect the pressure gradient. as shown equation (12)

$$\rho \frac{\partial w_1}{\partial t} + \rho w_1 \frac{\partial w_1}{\partial x} - \mu \frac{\partial^2 w_1}{\partial x^2} = 0. \quad (12)$$

If we use the conventional variable w instead of w_1 and define v as the kinematic viscosity, i.e., $\nu = \frac{\mu}{\rho}$, then the aforementioned equation is the viscid Burgers' equation, as given in (1). It is tempting to delete the second-derivative element from (9) when the fluid's viscosity μ is close to zero as an idealisation. Because of this, as shown equation (13)

$$\rho \frac{\partial w_1}{\partial t} + \rho w_1 \frac{\partial w_1}{\partial x} = 0, \quad (13)$$

This, as demonstrated in (2), becomes the inviscid Burgers' equation upon setting $w_1 := w$ and dividing by ρ. It turns out that extra physical constraints are needed to prevent equation (13) from obtaining physically nonsensical solutions if it is to be used as a model for the dynamics of an in-viscous fluid. Working with (13) is significantly simpler than working with (8), thus the extra length is well worth it.

2.5. Traffic flow

Let's say we're talking about traffic on a highway, and by "density" we mean $\rho(x,t)$, and by "traffic flow" we mean $f(x,t)$. In addition, we will take into account ρ^* the range $0 \le \rho^* \le \rho_{max}$ where $\rho - max$ is the value at which vehicles are literally touching each other on the road. Since the number of vehicles is a conserved quantity, the continuity equation may be used to connect automobile density to flow. as shown in equation (14)

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial f}{\partial x^*} = 0 \quad (14)$$

When considering a potential formulation for the flow, one's mind immediately jumps to $f = v\rho^*$, where v represents the velocity. However, it turns out that we need to assume that f is a function of the density gradient as well to represent the reality that drivers would slow down to accommodate for a rising density ahead. For the sake of argument, let's assume as shown equation (15)

$$f(\rho^*) = \rho^* v(\rho^*) - D \frac{\partial \rho^*}{\partial x^*}, \quad (15)$$

for any constant D. We also assume that v is a constant independent of ρ^*: We'd all prefer to travel at the maximum speed $v - max$ on the highway (the posted limit, maybe), but when traffic is heavy, we have to slow down. The most basic kind of connection to know this is as shown equation (16)

$$v(\rho^*) = \frac{v_{max}}{\rho_{max}} (\rho_{max} - \rho^*) \quad (16)$$

Putting (15) and (16) into (14) leads to as shown equation (17,18,19)

$$\frac{\partial \rho^*}{\partial t^*} + \frac{d}{dx^*}\left[\frac{v_{max}}{\rho_{max}}(\rho_{max} - \rho^*)\rho^*\right] = D\frac{\partial \rho^{*2}}{\partial x^{*2}}. \quad (17)$$

Scaling through

$$v_{max} = \frac{x_0}{t_0}, \rho = \rho_{max} \times \rho^*, x = x_0 \times x^* \quad (18)$$

and $t = t_0 \times t^*$ results in

$$\frac{\partial \rho}{\partial t} + \frac{\partial((1-\rho)\rho)}{\partial x} = \epsilon \frac{\partial^2 \rho}{\partial x^2} \quad (19)$$

with $\epsilon = \frac{D}{v_{max} \times x_0}$ and $0 \le \rho \le 1$. shown equation(20,21,22)

By transformation $w = 2\rho - 1$, we have as

$$\frac{\partial w}{\partial t} = 2\frac{\partial \rho}{\partial t}, \frac{\partial w}{\partial x} = 2\frac{\partial \rho}{\partial x}, \frac{\partial^2 w}{\partial x^2} = 2\frac{\partial^2 \rho}{\partial x^2} \quad (20)$$

Now, taking $u = \rho(1-\rho) = \rho - \rho^2$, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \rho}.\frac{\partial \rho}{\partial x} = (1-2\rho)\frac{\partial \rho}{\partial x} = -w\frac{\partial \rho}{\partial x} = -\frac{1}{2}w\frac{\partial w}{\partial x}. \quad (21)$$

Putting (19) and (20) in (21) imply

$$\frac{1}{2}\frac{\partial w}{\partial t} - \frac{1}{2}w\frac{\partial w}{\partial x} = \epsilon\frac{1}{2}\frac{\partial^2 w}{\partial x^2} \quad (22)$$

that results in the viscid Burgers' equation as it is indicated in (1).

2.6 Characteristic method

Here, we present the inviscid Burgers' equation solution with given initial value. To this end, we present the characteristic model. Before that, we express the following definition:

Definition 2. Let Ω be an open subset of $\mathbb{R}^n$. The space $C^k(\Omega)$ consists the set of all functions $f: \Omega \to \mathbb{R}$ that are k −times continuously differentiable.

Consider the first value problem as shown equation (23)

$$\begin{cases} \frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = 0, \ x \in \mathbb{R}, t > 0, \\ w(x,0) = w_0(x), \ x \in \mathbb{R}. \end{cases} \quad (23)$$

In order to solve (23) by characteristic method, we consider the following problem: as shown equation (24)

$$\begin{cases} x'(t) = w(x(t),t), \ \ t > 0, \\ x(0) = x_0. \end{cases} \quad (24)$$

If $x(t)$ and $w(x,t)$ ($\in C^1$) are solutions of (23) and (24) respectively, then as shown equation(25)

$$\frac{d}{dt}[w(x(t),t)] = \frac{\partial w(x(t),t)}{\partial t} + x'(t)\frac{\partial w(x(t),t)}{\partial x} = \frac{\partial w(x(t),t)}{\partial t} + w(x(t),t)\frac{\partial w(x(t),t)}{\partial x} = 0, \quad (25)$$

i.e, w is constant along the characteristic curve $x(t)$ and therefore as shown equation (26)

$$w(x(t),t) = w(x(0),0) = w_0(x_0), \quad (26)$$

which considering the system (23), suggests that the defining curves are straight lines whose shape is fixed by the beginning conditions. as shown equation (27)

$$x = x_0 + w_0(x_0)t, \quad t > 0. \quad (27)$$

In principle, one could invert (19) to obtain $x_0 = x_0(x,t)$. Then, using (26), one would obtain the solution $w(x,t) = w_0(x_0(x,t))$. Nevertheless, as the necessary inversion is often not solvable analytically, one can employ a symbolic calculation programme to build a separated solution to (24) by dragging the starting data $w_0(x_0)$ along the property line (27), as shown in the following example. Figure 1 shows the solution of w on the line. As shown in the figure (1).

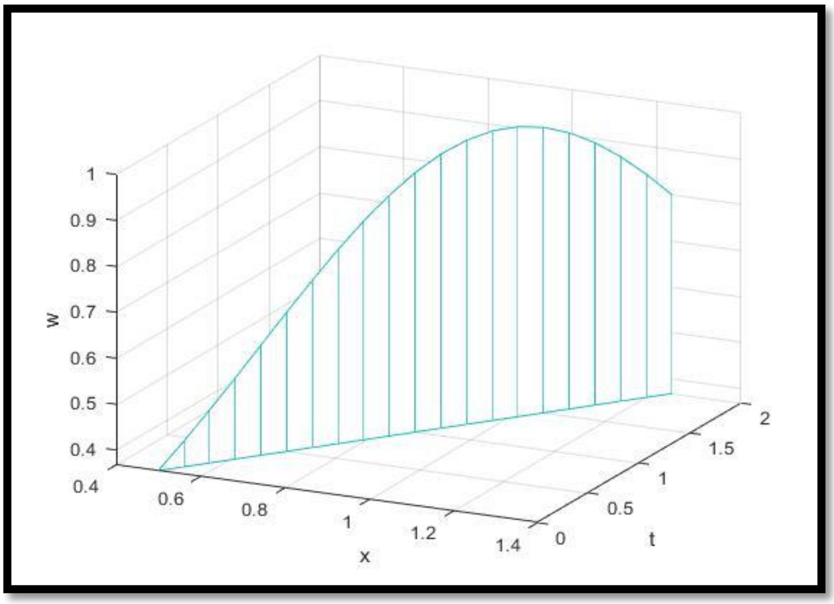

Figure (1): Output of the program for $w_0(x) = e^{-(2(x-1))^2}$ with initial value $x_0 = 0.5$

3.Methodology

3.1 Methodology solutions of Burgers' equation

Here, we outline a few techniques that can be used to solve the Burgers' equation [11].

Consider the Burgers' equation (1). Taking $\frac{\partial \psi}{\partial x} = w$ implies that as shown equation(28,29,30,31,32)

$$\frac{\partial^2 \psi}{\partial t \partial x} + \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial x^2} = \epsilon \frac{\partial^3 \psi}{\partial x^3}. \quad (28)$$

Integrating both side of (28) yields that

$$\frac{\partial \psi}{\partial t}+\frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2=\epsilon\frac{\partial^2 \psi}{\partial x^2} \quad (29)$$

Now, in view of Cole-Hopf transformation

$$\psi=-2\epsilon \ln \phi, \quad (30)$$

There are

$$w(x,t)=\frac{\partial \psi}{\partial x}=-2\epsilon\frac{\phi_x}{\phi}, \quad (31)$$

where $\phi_x=\frac{\partial \phi}{\partial x}$.

Putting (31) into (1) concludes that

$$\frac{\partial \phi}{\partial t}=\epsilon\frac{\partial^2 \phi}{\partial x^2}. \quad (32)$$

Then, using Cole-Hopf transformation, the nonlinear Burgers' equation (1) is converted to a simple linear equation.
Another technique that is a semi-analytical approach is based on Adomian decomposition technique that we express it in the sequel:

Take into account the Burgers' equation (1) with initial situation $w(x,0)=f(x)$. In other words, we consider the subsequent problem: as shown equation (33,34,35).

$$\begin{cases}\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial x}=\epsilon\frac{\partial^2 w}{\partial x^2}, \\ w(x,0)=f(x).\end{cases} \quad (33)$$

Applying the inverse operator

$$L_t^{-1}(.)=\int_0^t . \, d\tau, \quad (34)$$

into Burgers' equation (33) implies

$$L_t^{-1}\frac{\partial w}{\partial t}+L_t^{-1}\left(w\frac{\partial w}{\partial x}\right)=\epsilon L_t^{-1}\frac{\partial^2 w}{\partial x^2} \quad (35)$$

Now, by noting the initial condition, we have as shown equation(36,37)

$$w(x,t)=f(x)+\epsilon L_t^{-1}\frac{\partial^2 w}{\partial x^2}-L_t^{-1}\left(w\frac{\partial w}{\partial x}\right) \quad (36)$$

Suppose that

$$w(x,t)=\sum_{n=0}^{\infty} w_n(x,t). \quad (37)$$

In other words, the solution to (33), which we write as a function series. Putting (36) in (37) yields that as shown equation (38,39)

$$\sum_{n=0}^{\infty} w_n(x,t)=f(x)+\epsilon\, L_t^{-1}\left(\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} w_n(x,t)\right)\right)-L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \quad (38)$$

Now, considering the subsequent recursive equation

$$\begin{cases} w_0(x,t) = f(x,t), \\ w_{k+1}(x,t) = L_t^{-1}\left(\frac{\partial^2 w_k}{\partial x^2}\right) - L_t^{-1}(A_k) \end{cases} \quad (39)$$

solves the equation (39). On the other hand, by (1.29-1.31), we have as shown equation (40,41,42,43,44)

$$\sum_{n=0}^{\infty} A_n = \left(\sum_{n=0}^{\infty} w_n\right)\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} w_n\right) = \sum_{n=0}^{\infty} w_n \times \sum_{n=0}^{\infty} \frac{\partial w_n}{\partial x} \quad (40)$$

Now, Adomian polynomial terms A_n can be computed in the following manner:

$$A_0 = w_0 \frac{\partial w_0}{\partial x}, \quad (41)$$

$$A_1 = w_1 \frac{\partial w_0}{\partial x} + w_0 \frac{\partial w_1}{\partial x}, \quad (42)$$

$$A_2 = w_2 \frac{\partial w_0}{\partial x} + w_1 \frac{\partial w_1}{\partial x} + w_0 \frac{\partial w_2}{\partial x}, \quad (43)$$

$$A_3 = w_3 \frac{\partial w_0}{\partial x} + w_2 \frac{\partial w_1}{\partial x} + w_1 \frac{\partial w_2}{\partial x} + w_0 \frac{\partial w_3}{\partial x}. \quad (44)$$

This process is continued to the end. Putting the values of A_n from the above process into the recursive equation (39) , the solution w_k and finally w is derived.

Example 1. Solve the Burgers' equation as shown equation (45,46,47,48,49,50,51,52)

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = \epsilon \frac{\partial^2 w}{\partial x^2}, \quad (45)$$

with initial condition $w(x,0) = x$.

Solution: First we apply the decomposition method. We first apply the operator L_t^{-1} and use (38) as below:

$$\sum_{n=0}^{\infty} w_n(x,t) = x + L_t^{-1}\left(\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} w_n(x,t)\right)\right) - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \quad (46)$$

Now, equating w_n both side of (46) yields: $w_0(x,t) = x$,

$$w_{k+1}(x,t) = L_t^{-1}\left(\frac{\partial^2 w_k}{\partial x^2}\right) - L_t^{-1}(A_k), \quad k = 0, 1, \cdots \quad (47)$$

In view of Adomian polynomial, we have $w_0(x,t) = x$,

$$w_1(x,t) = L_t^{-1}\left(\frac{\partial^2 w_0}{\partial x^2}\right) - L_t^{-1}(A_0) = L_t^{-1}(0) - L_t^{-1}\left(w_0 \frac{\partial w_0}{\partial x}\right) = -\int_0^t x \times 1 d\tau = -xt, \quad (48)$$

$$w_2(x,t) = L_t^{-1}\left(\frac{\partial^2 w_1}{\partial x^2}\right) - L_t^{-1}(A_1) = L_t^{-1}(0) - L_t^{-1}\left(w_1 \frac{\partial w_0}{\partial x} + w_0 \frac{\partial w_1}{\partial x}\right) = 0 - L_t^{-1}\big(-xt \times 1 + x \times (-t)\big) = -L_t^{-1}(-2xt) = -\int_0^t -2x\tau d\tau = xt^2, \quad (49)$$

$$w_3(x,t) = L_t^{-1}\left(\frac{\partial^2 w_2}{\partial x^2}\right) - L_t^{-1}(A_2) = 0 - L_t^{-1}\left(xt^2 \times 1 + (-xt)\times(-t) + x \times t^2\right) = -L_t^{-1}\left(3xt^2\right) = -\int_0^t 3x\tau^2 d\tau = -xt^3 \quad (50)$$

$$w_n(x,t) = (-1)^n xt^n \quad n \geq 0 \quad (51)$$

Repeating this recursive process gives the following series of the solution

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) = x - xt + xt^2 - xt^3 + \cdots = x\left(1 - t + t^2 - t^3 + \cdots\right). \quad (52)$$

By geometry series, we know if $|t| < 1$, then

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots \quad (53)$$

This implies that

$$w(x,t) = \frac{x}{1+t}, \quad |t| < 1 \quad (54)$$

Example 2: Solve the Burgers's equation as shown equation (55,56,57)

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = \epsilon\frac{\partial^2 w}{\partial x^2} \quad (55)$$

with initial condition

$$w(x,0) = 1 - \frac{2}{x}, \ x > 0. \quad (56)$$

Solution: Again as the before example, using decomposition method, we have

$$\sum_{n=0}^{\infty} w_n(x,t) = 1 - \frac{2}{x} + L_t^{-1}\left(\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} w_n(x,t)\right)\right) - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \quad (57)$$

The following recursive relation is deduced by (57) : as shown equation(58,59,60,61,62,63,64)

$$w_0(x,t) = 1 - \frac{2}{x}, \quad (58)$$

$$w_{k+1}(x,t) = L_t^{-1}\left(\frac{\partial^2 w_k}{\partial x^2}\right) - L_t^{-1}(A_k), \ k = 0, 1, \cdots \quad (59)$$

By expanding this recursive equation, we have

$$w_0(x,t) = 1 - \frac{2}{x}, \quad (60)$$

$$w_1(x,t) = L_t^{-1}\left(\frac{\partial^2 w_0}{\partial x^2}\right) - L_t^{-1}(A_0) = -\frac{2}{x^2}t, \quad (61)$$

$$w_2(x,t) = L_t^{-1}\left(\frac{\partial^2 w_1}{\partial x^2}\right) - L_t^{-1}(A_1) = -\frac{2}{x^3}t^2, \quad (62)$$

$$w_3(x,t) = L_t^{-1}\left(\frac{\partial^2 w_2}{\partial x^2}\right) - L_t^{-1}(A_2) = -\frac{2}{x^4}t^3, \quad (63)$$

$$w_n(x,t) = -\frac{2}{x^{n+1}}t^n. \ n \geq 1 \quad (64)$$

So,

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) = 1 - \sum_{n=0}^{\infty} \frac{2}{x^{n+1}} t^n = 1 - \frac{2}{x} \sum_{n=0}^{\infty} \left(\frac{t}{x}\right)^n. \qquad (65)$$

By geometry series formula, if $|a| < 1$, we have as shown equation (66,67)

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}. \qquad (66)$$

This gives the final solution:

$$w(x,t) = 1 - \frac{2}{x}\left(\frac{1}{1-\frac{t}{x}}\right) = 1 + \frac{2}{t-x}. \qquad (67)$$

Conclusions

In conclusion, it can be said that the Burgers time equation represents an important topic in many scientific and technical fields. Various forms of the basic Burgers equation are studied and its general form is presented. The fractional time form of the Burgers equation is developed and also presented. The literature on the Burgers equation and its various forms has been reviewed. There is also a review of converting the Burgers equation to the Burgers model. It can be concluded that investigating analytical solutions of the Burgers equation is challenging due to its nonlinear nature and low viscosity. Therefore, numerical methods are very important in finding solutions to this equation. The results of this research can be used in diverse fields such as traffic characterization, heat distribution and gas dynamics. Understanding and analyzing the Burgers time equation and its applications are of great importance to researchers and engineers in these disciplines.

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